المملكة العربية السعودية
وزارة التعليم العالي
جامعة أم القرى
كلية العلوم التطبيقية
قسم الفيزياء

خصائص مسارات الأشعة في الأوساط الغير متجانسة

رسالة مقدمة إلى قسم الفيزياء بكلية العلوم التطبيقية بجامعة أم القرى
كمتطلب تكميلي لبرنامج درجة الماجستير في الفيزياء

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الفصل الدراسي الثاني

١٤٢٤/٢٠٠٣

الإهـــــداء

لأمي الحبيبــــة ..

لزوجي الغالـــــي..

لأولادي (عبدالعزيز و عبد المجيد)..

هند

الشكر و العرفان

أود أن أتقدم بخالص شكري و تقديري لكل من سعادة الدكتور/ عصام بن حامد الأهدلي ،المشرف على الرسالة، و سعادة الدكتور/ عبد محمد قاسم ،مساعد المشرف، اللذان أفادا بحثي بعلمهما فلم يبخلا عليّ بوقتهما الثمين على الرد على جميع أسئلتي و مناقشتي المناقشة العلمية . فالشكر كل الشكر لهما على دعمهم و توجيههم و تشجيعهم لي خلال إعداد هذا البحث.

كما أتقدم بالشكر الجزيل لزوجي العزيز على صبره، مساعدته، وتشجيعه المستمر.

ملخص البحث

الوسط الغير متجانس يتغير بداخله معامل الانكسار من نقطة إلى أخرى. في الوقت الحاضر، معامل الانكسار المتحدر (gradient index, GRIN) هو المستخدم عادة في وصف مثل هذه الأوساط. إن إمكانية استخدام هذه الأوساط في الوسائل البصرية المصنعة يدوياً اُفترِضت من عدة سنوات، و لكن الآن من الممكن استخدامها لتصنيع المواد المناسبة لاستغلال هذه الفكرة. و قد كُرست العديد من الأبحاث في هذا المجال ليس فقط للتوسع النظري و إنما لتطوير طرق الإنتاج و قياس المواد الضرورية.

إن صناعة الاتصالات طورت استخدام ألياف الـGRIN كوسائل نقل عالية الكفاءة، حيث يلاحظ إن الضوء المفقود في ألياف الـGRIN يميل لأن يكون منخفض جداً، في حين أن الأشعة داخل هذه الألياف تنحني بطريقة لا تنعكس فيها على أسطح الألياف.

في العقدين الأخيرين، كُرست مجموعة الدراسات على بصريات الـ GRIN فعلى سبيل المثال تطبيق تدرجات معامل الانكسار في الأنظمة البصرية الكلاسيكية ذات العدسات منحدرة معامل الانكسار قد حسنت نتائج التطبيق سواءً باختزال العناصر العددية للعدسات و استبدال الأسطح المنحنية بالعناصر المتدرجة.

إن سؤال كيفية تحديد المسارات (عادةً منحنية) للأشعة الممكنة و المتنوعة في وسط الـGRIN أُجيب عنه سابقاً بصورة مبدئية حيث يمكن أن يعبر عن النتائج على هيئة معادلة تفاضلية متجهة مفردة من الدرجة الثانية (معادلة الشعاع)، وهذه المعادلة مستنتجه من مبدأ فِرمات(Fermat). إن الحل التقريبي لهذه المعادلة يعطي أشعة محورية(قريبة من المحور البصري) و في بعض الحالات الخاصة لل GRIN أعطت

الأشكال له. كما ان التقنية الرقمية الحاسوبية فيما بعد طورت تتبع اثر الأشعة خلال وسط الـGRIN. في هذه الدراسة، هذه التقنية سوف نستخدمها في دراستنا في هذا البحث لمعرفة من خصائص الشعاع داخل وسط الـ GRIN.

بسم الله الرحمن الرحيم

وزارة التعليم العالي
جامعة أم القرى
كلية العلوم التطبيقية

غوذج رقم (٨)

إجازة أطروحة علمية في صيغتها النهائية بعد إجراء التعديلات

الاسم (رباعي) : .... كلية : العلوم التطبيقية قسم : ....

الأطروحة مقدمة لنيل درجة : .... في تخصص : ....

عنوان الأطروحة : ((........................

الحمد لله رب العالمين والصلاة والسلام على أشرف الأنبياء والمرسلين وعلى آله وصحبه أجمعين وبعد :

فبناءً على توصية اللجنة المكونة لمناقشة الأطروحة المذكورة أعلاه والتي تمت مناقشتها بتاريخ ١٤٢٥هـ بقبولها بعد إجراء التعديلات المطلوبة، وحيث قد تم عمل اللازم ؛ فإن اللجنة توصي بإجازتها في صيغتها النهائية المرفقة للدرجة العلمية المذكورة أعلاه ...

والله الموفق ...

أعضاء اللجنة

المشرف	المناقش الداخلي	المناقش الخارجي
الاسم :	الاسم :	الاسم : ....
التوقيع :	التوقيع :	التوقيع : ....

يعتمد

رئيس قسم

الاسم : د/ صبري بن أسعد أبو فطور

التوقيع : ....

• يوضع هذا النموذج أمام الصفحة المقابلة لصفحة عنوان الأطروحة في كل نسخة من الرسالة .

تاريخ الاستلام : ٢٩/٢/١٤٢٥هـ
المستلم أ.د صبري أسعد أبو فطور

Kingdom of Saudi Arabia
Ministry of Higher Education
Umm Al Qura University

THE CHARACTERISTICS OF RAYS PATHS IN A GRIN MEDIUM

By

Hind Abdul Aziz Al Hajjaji

Supervised by

Dr. I. H. Al Ahdali and Dr. A. M. Kassim

A thesis submitted in partial fulfillment
of the requirements for the degree of
master of science
(Physics)

Physics Department
Faculty of Applied Sciences
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ABSTRACT

A nonhomogeneous medium is one in which the refractive index varies from point to point within the medium. Currently the term "gradient index" (GRIN) is often used to describe such medium. The possibility of using this medium in man-made optical instruments has been considered for many years, but now it has been possible to fabricate the appropriate materials to exploit this idea. Many researches have been devoted not only to theoretical developments in this field, but also to improved methods of producing and measuring the necessary materials.

The communication industry has been developing the use of GRIN fibers as a highly efficient means of transporting information. Light loss in GRIN fiber tend to be very low, since rays within this fiber are curved in such a way they are not reflected at the fiber surfaces.

In the last two decades, an extensive literature was devoted to GRIN optics, i.e., the application of gradients in optical systems of classical types: gradient index lenses. It has been shown that improved results can be achieved by gradients in this application; either through aspheric surfaces can be replaced by gradient elements.

The question of determining the exact paths (usually curved) of the various possible rays in GRIN medium was answered, in principle, long ago. The result can be expressed in the form of a single vectorial differential equation of second order (Ray Equation), an equation, which can be deduced from Fermat's principle. In the literature, an approximated solution for this equation has been given for a paraxial rays (near the optical axis of the system) and for some special cases of GRIN profiles. Later on, a numerical technique has been developed to trace rays through a GRIN medium, using a computational program. In this study, this technique will be used to investigate the characteristics of the ray path inside GRIN media.

ACKNOWLEDGEMENTS

I would like to express my deep gratitude to both Dr. Issam Hamed Al Ahdali and Dr. Abed Mohammed Kassim, for introducing me to the subject of this research and providing active continuous supervision during the course of this work. Their valuable guidance and discussions are highly appreciated.

I wish to thank my mother for her encouragement during my study. My special thanks and prayers go to my father for every thing he did for me before his death that I will never forget. May Allah rest his soul in peace.

Finally, I gratefully acknowledge the patience, assistance, encouragement, and understanding of my faithful husband. Without his help, I would never have succeeded.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

INTRODUCTION

Virtually all optical glass used today consists of a homogeneous(single index), isotropic blend of materials[32]. The index of refraction therefore only varies with wavelength, and not with position within the blank.

In nonhomogeneous medium the refractive index varies from point to another within the medium. Currently the terms "gradient index" and "graded index" are often used to describe such media. A nonhomogeneous media occur frequently in nature. Examples are the lens of the human eye and the atmosphere of the earth. The possibility of using such media in man-made optical instruments has been considered for many years [16], but only recently

has it been possible to fabricate the appropriate materials to exploit this idea.

In view of the impressive advances in the field of homogeneous fiber optics within the past twenty years[16], it is natural that the idea of making gradient index fibers should be actively explored. This has, indeed, taken place, and already endoscopes made for the medical profession by this means are commercially available. The chief advantage of gradient fiber in this application is that a single fiber instead of bundle of fibers can be used to transmit an image. The communication industry has been developing the use of gradient index fibers as a highly efficient means of transporting information. Light losses in a gradient fiber tend to be very low, since rays within a fiber are curved in such a way that they are not reflected at the surfaces of the fiber.

For many years optical designers have been intuitively aware of the value and potential that an inhomogeneous refractive index distribution can bring

to manufacture of precision optical instruments. Even so , designers have been cautious when considering lens designs with inhomogeneous glass , partly because of design difficulties, but mostly because of the need for controlled and reliable materials[3].

In 1100 A.D, Al-Hazzan[16] suggested that astronomical refraction was responsible for the flatting of the sun's disk near the horizon. In 1656, Cassini[16] assumed that the atmosphere is inhomogeneous shells so there is a continuously varying refraction index having spherical symmetry about the earth. In 1854, Maxwell[16], described a medium involves an index function having spherical symmetry about a point and would be expected to have the form of a sphere. In 1905, Wood[5] describe a method of a simple lens having two plane surfaces and refractive index varying with the distance from the optical axis. The first patent on progressive addition lens was given to Aves [5] in 1907. The power gradient was due to both lens surfaces being shaped so

the radius of curvature in the horizontal direction varied along the lens vertical.

In 1920, a French patent[5] was issued for a lens which required only asingle surface figure to provide the power progression. The earliest reference of using a gradient index materials (GRIN) in corrective lenses is the 1924 patent[5],which describes a method to make a bifocal lens by using a gradient index glass. An other form appears in 1946, Beach patent[5]. It describes a lens with front surface having both a near power region and a far power region connected by annular zones of power progression.

In 1964, Lunberug [31] discovered a lens has an index function with spherical symmetry about a point.

The ground work for the analytical treatment and understanding of nonhomogeneous glass was developed in the early seventies by Sands in a series of papers. In 1973, Sands[25] derived the expressions which can be used to compute the derivatives of its paraxial coefficients with respect to each surface curvature,

refractance, the thickness of media, and parameter specifying the index distribution.

In 1982, Sharma[26] and others represented a method for tracing ray through graded-index medium, consists of transforming the ray equation into a convenient form and solving the resulting using a standard numerical technique.

The goal of this research is to investigate characteristics of rays paths in GRIN lens (axial and radial) for different field angles and different index profiles by using Sharma's method[26].

In the following chapters, a theoretical background will be introduced for GRIN medium, from where how to fabricate and what the type of it is. Also , the ray equation will be derived in a general form, and the approximate solutions of this equation in different medium such as axial and radial GRIN will be solved.

More over the ray tracing will be represented through an axial GRIN medium step by step inside a plano-convex lens at different angles, and the effect of gradient parameter on

the ray tracing at different values are studied. Then computer programs which trace rays in non homogeneous medium such as, axial GRIN (linear and quadratic) and radial GRIN are presented. These programs are based on Sharma's method. Finally , ZEMAX[36] program is used to compute the optical aberrations for such optical system. The aberration comparisons have been made for homogeneous and nonhomogeneous lens to clearfy the usefull of GRIN for optimizing such optical medium system.

Chapter 2

THEORETICAL BACKGROUND

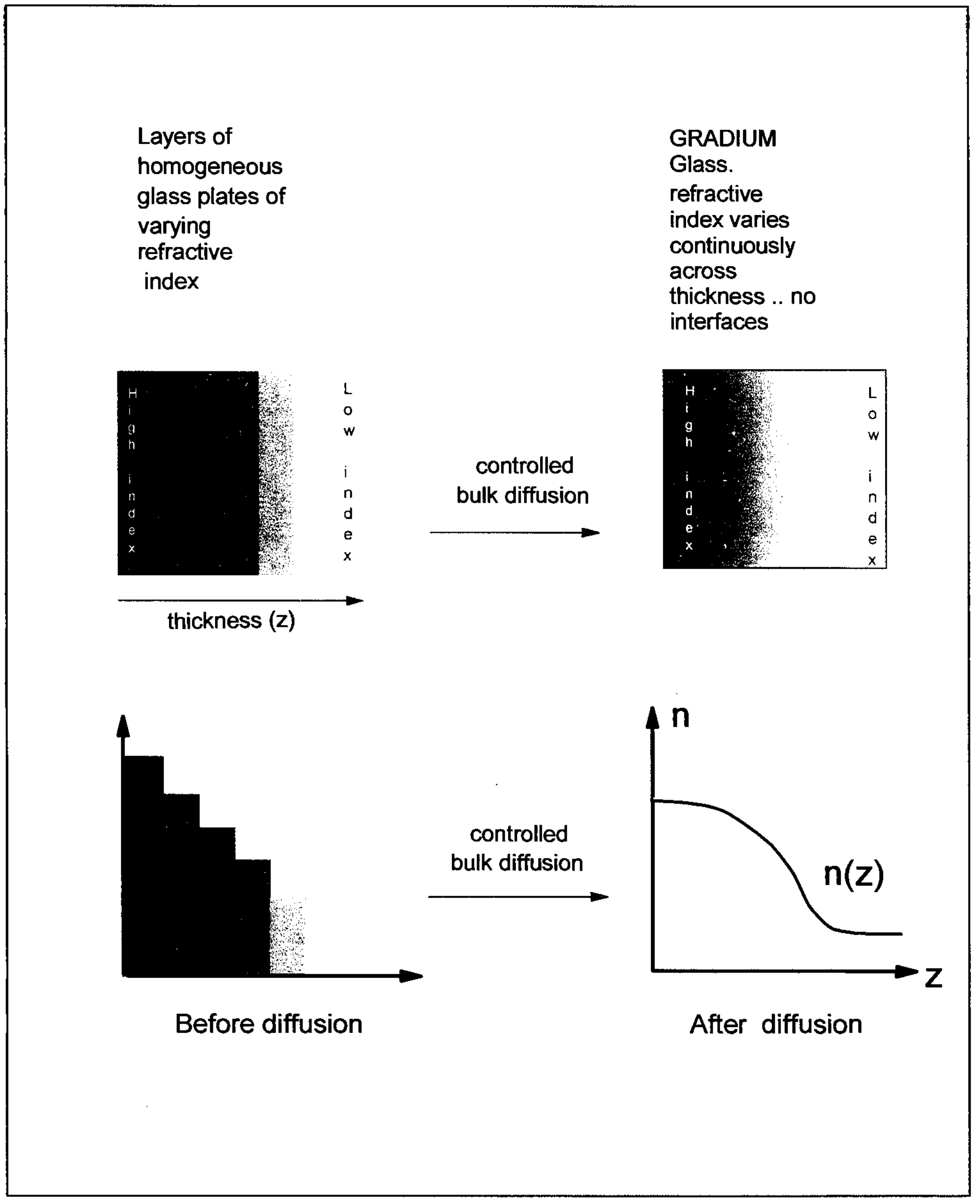
In this chapter we are going to present the concept of a GRIN medium, how to fabricate it and its types. Then, the derivation of ray equation in a general form will be presented. Also the solution of this equation in special cases, will be given.

2.1 *GRIN Manufacturing Technology, and Classification:*

The manufacturing technology of gradient-index materials (glass and plastic[13,22,25]) have a variety of ways to fabricate . In GRIN glasses, there are different fabrication methods which

have different index gradient size and depth, such as: ***Diffusion method***[16] and ***Crystal growing method***[5]. In a ***Diffusion method***, a base glass is immersed in a molten salt bath. Cations from the bath exchange with the alkali ions in the salt (composing the glass). This is a proven technology, with decades of study, and is used for commercial products. Also, design-for-manufacture methods are becoming available. However, it requires high-temperature ovens to melt the salt bath, and so is energy intensive,(see Fig.2.1). In a ***Crystal growing method***[5], a seed crystal is used to pull a large crystal from a bath containing a mixture of chemicals. As the crystal is pulled, the amount of one chemical in the bath decreases. This cause the crystal to take more of a second chemical, creating a change in refractive index. This is potentially useful for working with infrared optics.

The classification of gradient-index lens designed have some degree of symmetry, therefore there are many type of GRIN medium such as spherical, axial and radial GRIN. The earliest gradient-index design is a spherically symmetric



Figuer (2.1):Schematic of layer diffusion process for making GRADIUM glasses.

lens (Spherical GRIN)[5], (see Fig.2.2). The examples of this kind of lens are Lunberg[30] and Maxwell fisheye lens[16], the ideal Luneberg lens represents a spherical region with index of refraction increasing from the surface of the sphere toward the center of the sphere, (such as illustrated in Fig.2.2). The optical behavior of this spherical structure depends only on the index gradient under the normal assumed condition for the lens, namely that the lens is embedded in a dielectric medium whose index of refraction matches the index of refraction at the outer surface of the sphere. In this sense, the light beam is propagation in a uniform optical materials untill it encounters the index gradient region with a smooth (non-abrupt) transition from the uniform to gradient region of the refractive index. Prior to the emergence of the Luneberg lens, another lens structure with similar construction but different index gradient was presented by Maxwell and is called Maxwell's fisheye lens.

In the second type of GRIN lens, there are a very large and continuos change in refractive index in one dimension[33], (see

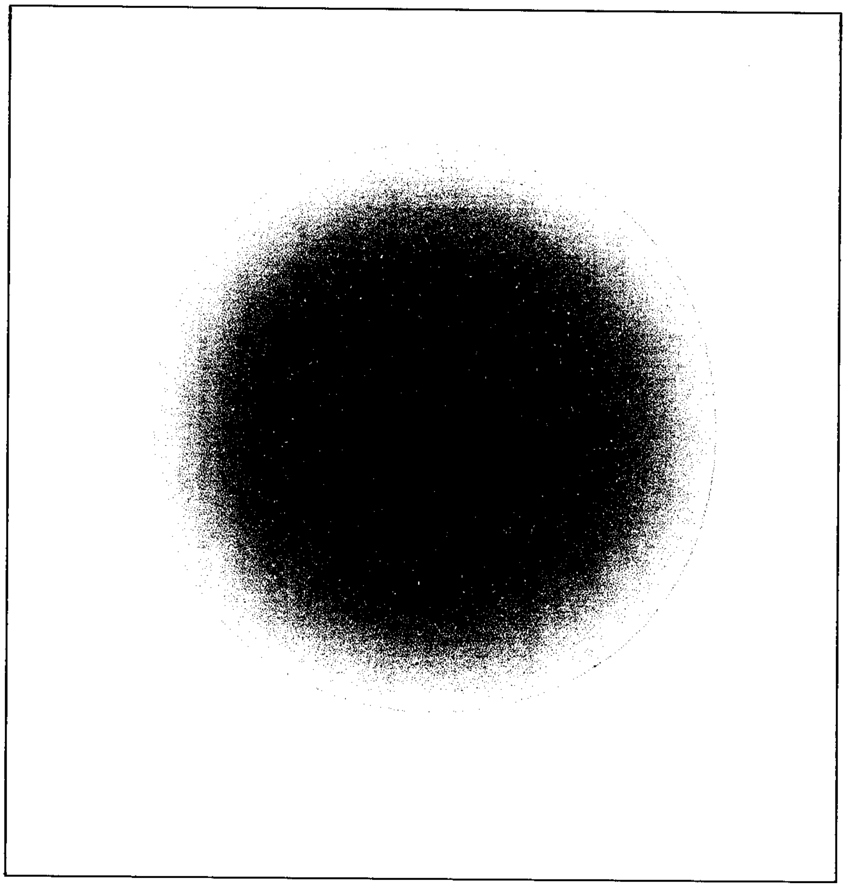


Fig.(2.2): Spherical gradient-index lens

Fig.2.3). Traditionally, the one dimensional refractive index gradient is called axial gradient, and most lenses are created with the gradient axis parallel to the optical axis of the lens. In this case, there is no inherent optical power (focusing capability) from the axial gradient. With axial GRIN lenses, a very useful effect is achieved when there is an axial gradient across the sag of a lens. In this case, the index variation along the lens sag is used to precisely tailor the angle of the refraction along the lens sag, thereby counteracting aberrations such as spherical, coma, astigmatism, distortion, field curvature, and cylindrical(cylindrical lenses)[17]-as well as complex combinations of aberrations [32]. Axial-gradient has become an appropriate choice of mainstream optical design because of the success of light-path's efforts improve the process to the point that repeatability and quality are excellent. However, despite the advantages to the optical design, there are many impediments to the designer. In particular, designing with inhomogeneous materials, the empirical experience base "bag of tricks" is smaller. As Sands[11] and others have pointed out, the

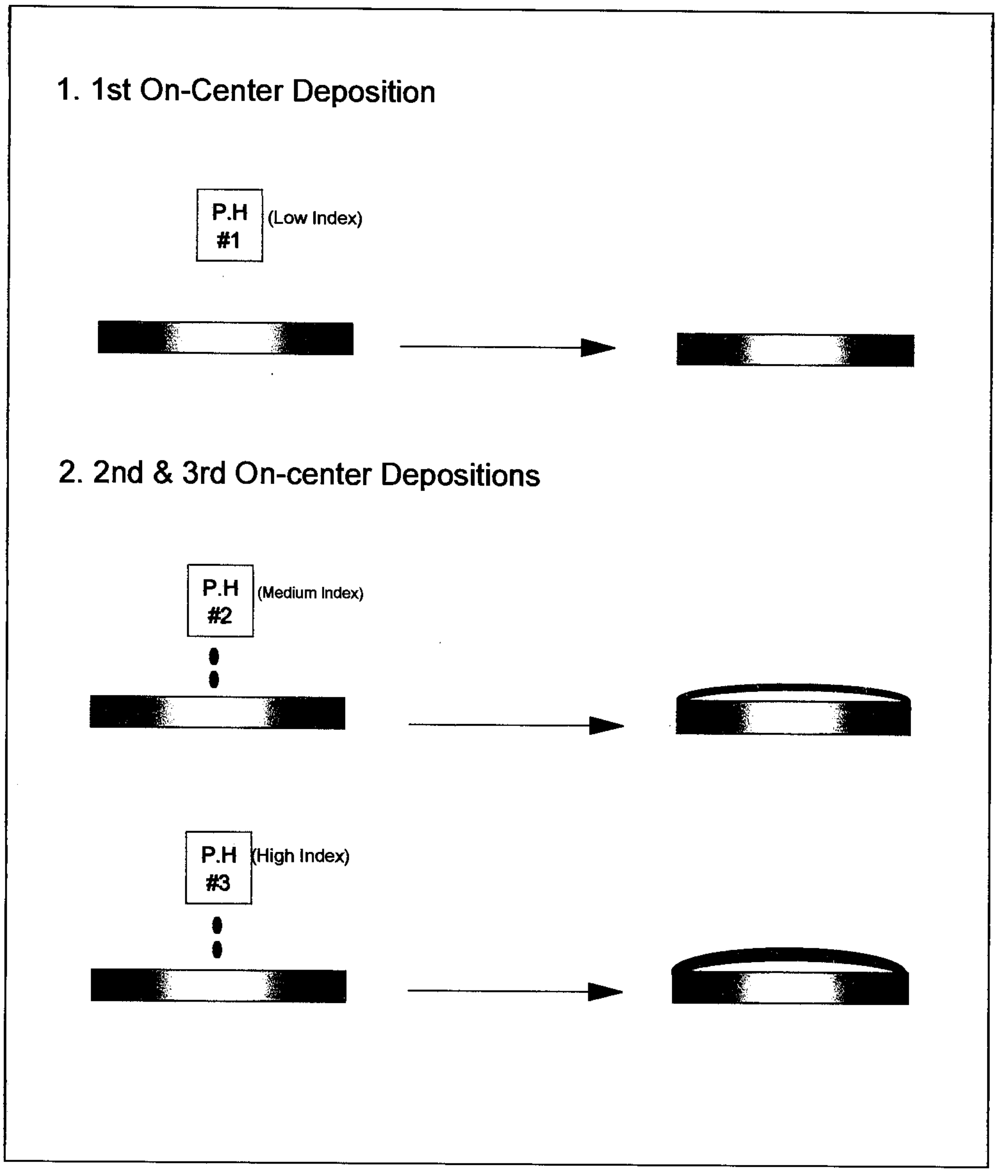


Fig.(2.3):Printing steps for axial GRIN lenses

inhomogeneous surface contribution of an axial gradient is functionally equivalent to that of an aspheric surface, at least to third order[14].In a third type, an index distribution having cylindrical symmetry, and the refractive index being a function of the radial distance from optical axis, therefor its called the radial gradient[16] ,(see Fig.2.4), the classic example of this type is the wood lens[16], (see Fig. 2.5).

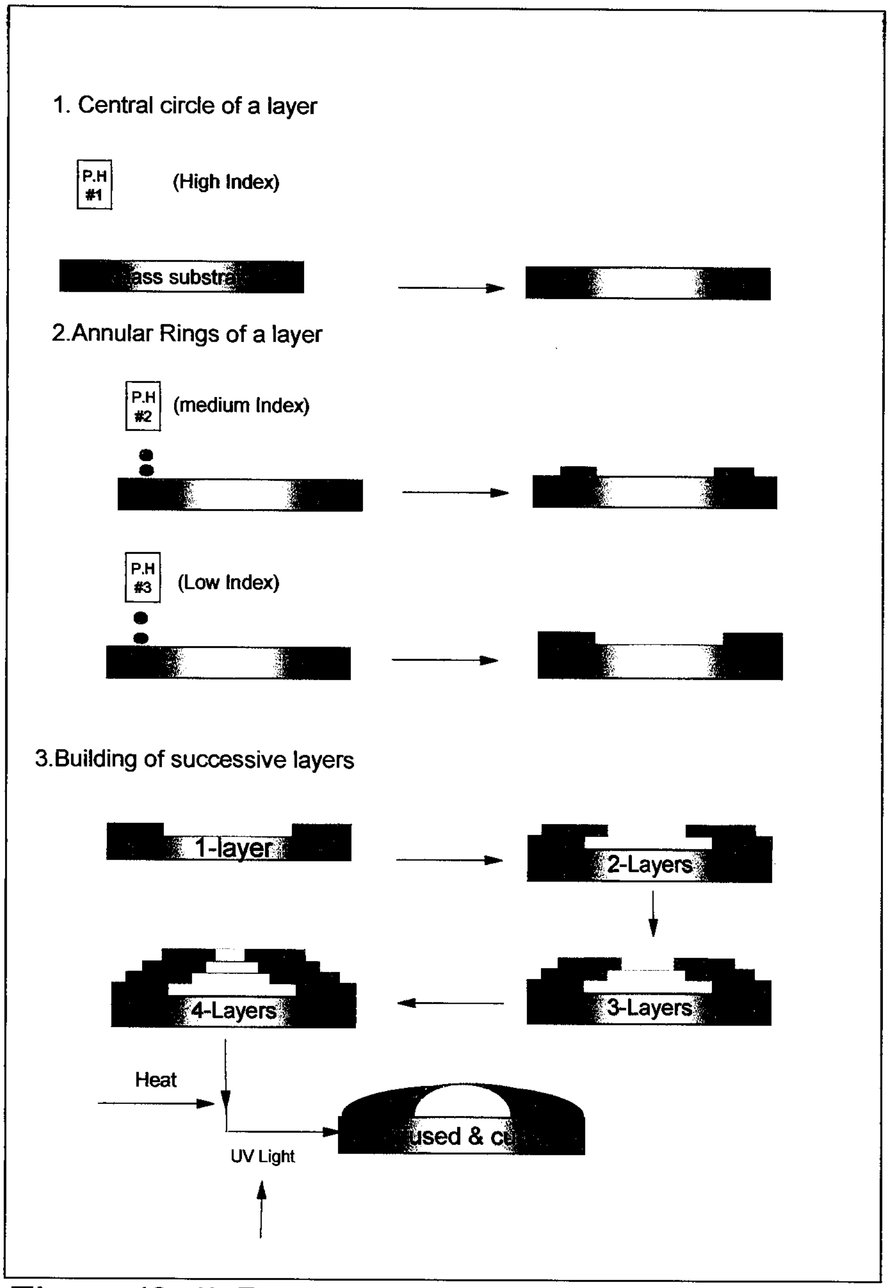
In next section we are going to derive the general form of ray equation. The derivation is based on Fermat principle[19] and by using the Lagrangian function[27].

2.2 *Derivation of Ray Equation*:

Consider a curve C which connects two points P and Q, (see Fig.2.6). The optical path length, S, along this curve is given as:

$$S = \int_{P}^{Q} n(x, y, z) ds \quad , \qquad (2.1)$$

where, ds is the infinitesimal arc-length[9] of the curve, and $n(x, y, z)$ is the refractive index function.



Figure(2.4):Process steps for radial GRIN lenses.

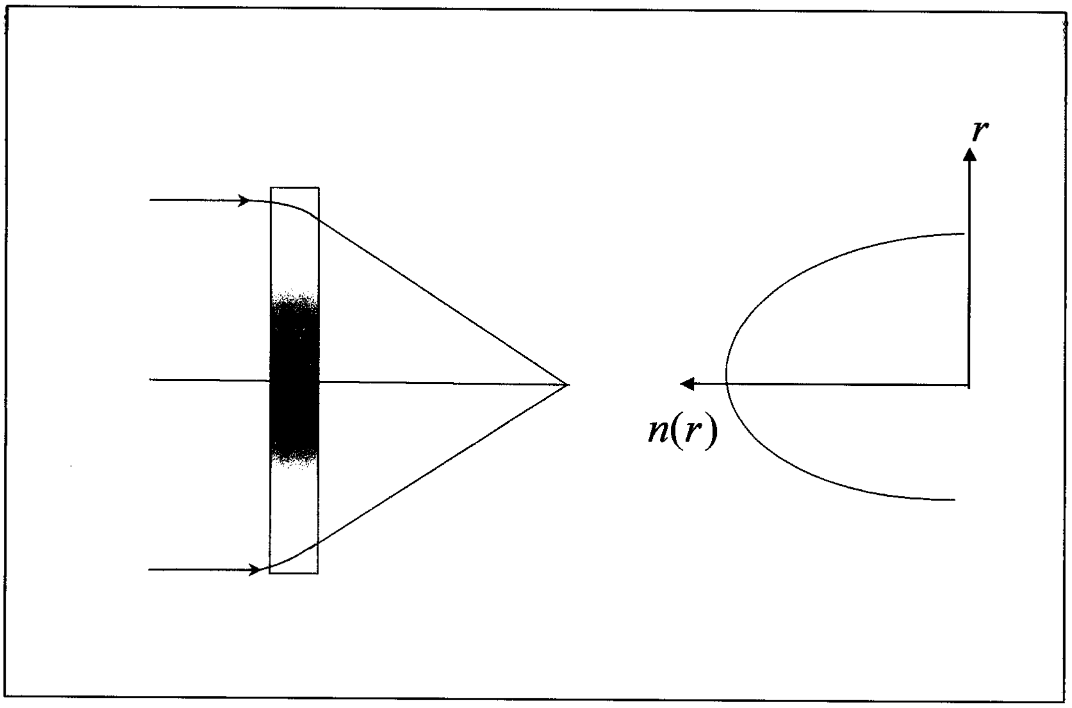


Fig.(2.5): A Wood lens with planer surfaces and radially-symmetric index gradient. The shaded region represents the index gradient, with darker regions indicating higher refractive index.

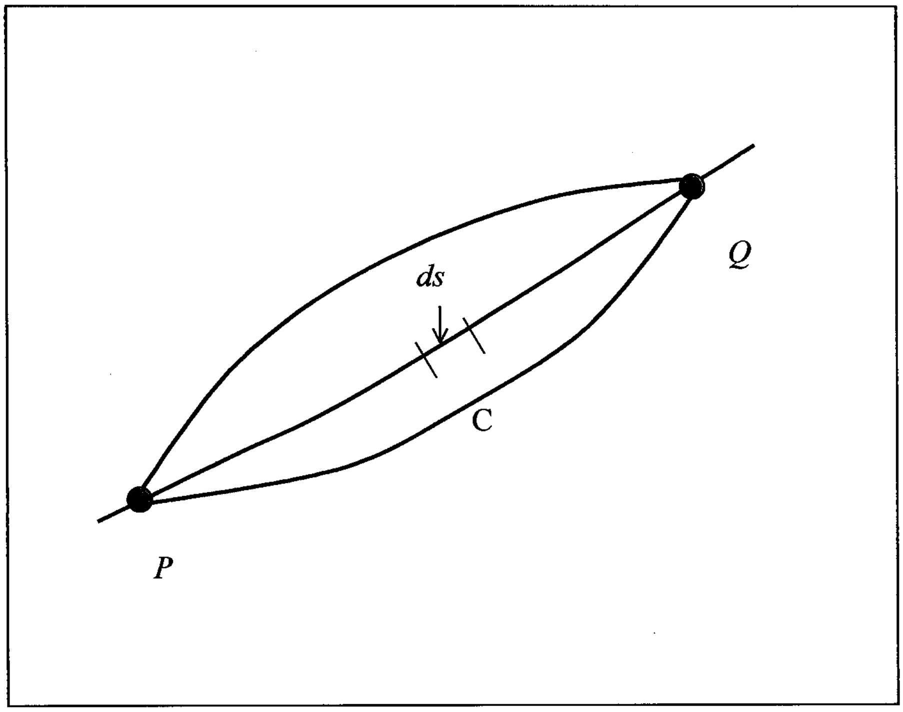


Fig.(2.6): The concept of arc - length.

Fermat principle [7,15]: if there are many paths which connect P and Q, then the light ray would follow that path for which the optical path length between the two points is an extremum (maximum, minimum, or stationary)[27], mathematically this principle is expressed as:

$$\delta \int_P^Q n(x,y,z)ds = 0 \tag{2.2}$$

where δ is the variation of integral.

In classical mechanics principle of least variation is given as:

$$\delta \int_{t_1}^{t_2} L dt = 0 \tag{2.3}$$

where L is called the Lagrangian function[9].

Mathematically the infinitesimal arc-length , ds, is given as:

$$ds = [(dx)^2 + (dy)^2 + (dz)^2]^{1/2} = dz[\dot{x}^2 + \dot{y}^2 + 1]^{1/2} \tag{2.4}$$

where dots represent differentiation with respect to z variable .

Substituting Eq.(2.4) in Eq.(2.2) gives:

$$\delta \int_P^Q n(x,y,z)(1 + \dot{x}^2 + \dot{y}^2)^{1/2} dz = 0 \tag{2.5}$$

Comparing Eq.(2.3) with Eq.(2.5), gives :

$$L(x,y,z,\dot{x},\dot{y},\dot{z}) = n(x,y,z)(1+\dot{x}^2+\dot{y}^2)^{1/2} \tag{2.6}$$

where $L(x,y,z,\dot{x},\dot{y},\dot{z})$ is called optical Lagrangian function(7).

Comparing with Lagrangian classical mechanics, the optical Lagrangian equations can be written as:

$$\frac{d}{dz}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \tag{2.7a}$$

$$\frac{d}{dz}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y} \tag{2.7b}$$

$$\frac{d}{dz}\left(\frac{\partial L}{\partial \dot{z}}\right) = \frac{\partial L}{\partial z} \tag{2.7c}$$

using Eq.(2.7a), the formula Eq.(2.6) gives:

$$\frac{d}{dz}\left[\frac{n\dot{x}}{(1+\dot{x}^2+\dot{y}^2)^{1/2}}\right] = (1+\dot{x}^2+\dot{y}^2)^{1/2}\frac{\partial n}{\partial x}$$

therefore:

$$\frac{\partial n}{\partial x} = \frac{1}{(1+\dot{x}^2+\dot{y}^2)^{1/2}}\frac{d}{dz}\left[\frac{n\dot{x}}{(1+\dot{x}^2+\dot{y}^2)^{1/2}}\right] \tag{2.8a}$$

simillarly,

$$\frac{\partial n}{\partial y} = \frac{1}{(1+\dot{x}^2+\dot{y}^2)^{1/2}}\frac{d}{dz}\left[\frac{n\dot{y}}{(1+\dot{x}^2+\dot{y}^2)^{1/2}}\right] \tag{2.8b}$$

$$\frac{\partial n}{\partial z} = \frac{1}{(1+\dot{x}^2+\dot{y}^2)^{1/2}}\frac{d}{dz}\left[\frac{n\dot{z}}{(1+\dot{x}^2+\dot{y}^2)^{1/2}}\right]. \tag{2.8c}$$

the formula Eq.(2.4) can be written as:

$$\frac{1}{(1+\dot{x}^2+\dot{y}^2)^{1/2}}\frac{d}{dz}=\frac{d}{ds} \qquad (2.9)$$

By using Eq. (2.9), Eq. (2.8a) can be written as:

$$\frac{\partial n}{\partial x}=\frac{d}{ds}\left[\frac{n\dot{x}}{(1+\dot{x}^2+\dot{y}^2)^{1/2}}\right]$$

since $\dot{x}=\frac{dx}{dz}$,then

$$\frac{\partial n}{\partial x}=\frac{d}{ds}\left[n\frac{dx}{dz(1+\dot{x}^2+\dot{y}^2)^{1/2}}\right] \qquad (2.10)$$

combaining Eq.(2.4) with Eq.(2.10) gives:

$$\frac{\partial n}{\partial x}=\frac{d}{ds}(n\frac{dx}{ds}) \qquad (2.11a)$$

Similarly the y-and z-equation can be written as:

$$\frac{\partial n}{\partial y}=\frac{d}{ds}(n\frac{dy}{ds}) \qquad (2.11b)$$

$$\frac{\partial n}{\partial z}=\frac{d}{ds}(n\frac{dz}{ds}) \qquad (2.11c)$$

Eqs.(2.11a-c) can be consider in a vectorial form as:

$$\frac{d}{ds}[n(\boldsymbol{r})\frac{d\boldsymbol{r}}{ds}]=\nabla n(\boldsymbol{r}) \qquad ,\nabla=\frac{\boldsymbol{r}}{r}\frac{d}{dr} \qquad (2.12)$$

Equation (2.12) is called ***Ray Equation*** ,where $\boldsymbol{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector of a point along the ray path through the medium.

As a special case the ray equation for a paraxial ray[18] (the ray makes a small angle with the z-axis ,optical axis of the system) , where $ds \approx dz$,can be written as:

$$\frac{d}{dz}(n\frac{d\boldsymbol{r}}{dz}) \approx \nabla n. \qquad (2.13)$$

In next section we will solve the ray equation in a homogeneous medium where the refractive index is fixed.

2.3 *Solution of ray equation in a homogeneous medium:*

In this type, the refractive index n is constant. Therefore $\nabla n = 0$, then Eq.(2.12) gives:

$$\frac{d^2\boldsymbol{r}}{ds^2} = 0 \qquad (2.14)$$

$$\frac{d\boldsymbol{r}}{ds} = \boldsymbol{a}$$

$$\boldsymbol{r} = s\boldsymbol{a} + \boldsymbol{b} \qquad (2.15)$$

where s is the arc length along the ray path. ***a*** and ***b*** are a constant vectors[30].

Eq.(2.15) is a vector equation of a straight line in the direction of the vector ***a***, passing through the point $\boldsymbol{r} = \boldsymbol{b}$. Hence in a homogeneous medium the paths of rays are straight lines[1].

In next section we are going to give the solutions of the ray equation in a nonhomogeneous medium.

2.4 *Solutions of ray equations in nonhomogeneous medium:*

In a nonhomogeneous medium such as an axial GRIN and a radial GRIN , the refractive index varies from one point to another in this medium. For example, in an axial GRIN medium, the refractive index varies with a distance along the optical axis 'z', i.e., it can be expressed as:

$$n(z) = N_{00} + N_{01}z + N_{02}z^2 + N_{03}z^3 + \dots \qquad (2.16a)$$

while in a radial GRIN medium, the refractive index varies with the radial distance from the optical axis 'r' , i.e., it can be expressed as:

$$n(r) = N_{00} + N_{10}r^2 + N_{20}r^4 + N_{30}r^6 + \ldots \qquad (2.16b)$$

Eqs. (2.16a-b) can be rewritten in a summation form as:

$$n(r,z) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} N_{ij}\, r^{2i}\, z^j \qquad (2.17)$$

where: r is the radial distance, z is the distance along the optical axis, n is the refractive index, and N_{ij} are real numerical constants which may be positive or negative.

According to Sharma's method [26,29] ,the solution of ray equation in nonhomogeneous medium, , can be summarized through the following steps:

Define a new variable t as:

$$t = \int \frac{ds}{n}; \qquad \text{or} \qquad dt = \frac{ds}{n} \qquad (2.18)$$

then, $\dfrac{d}{ds} = \dfrac{d}{dt}\dfrac{dt}{ds} = \dfrac{1}{n}\dfrac{d}{dt}$

By substituting Eq.(2.18) into Eq.(2.12) we have:

$$\frac{d}{n(\boldsymbol{r})dt}[n(\boldsymbol{r})\frac{d\boldsymbol{r}}{n(\boldsymbol{r})dt}] = \nabla n(\boldsymbol{r})$$

or

$$\frac{d}{dt}(\frac{d\boldsymbol{r}}{dt}) = n(\boldsymbol{r})\nabla n(\boldsymbol{r}) \qquad (2.19a)$$

since, $\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi$, where φ and ψ are scalar functions, and if $\varphi = \psi = n(\boldsymbol{r})$, then:

$$n(\boldsymbol{r})\nabla n(\boldsymbol{r}) = \frac{1}{2}\nabla[n^2(\boldsymbol{r})]$$

therefore Eq.(2.19a) reads:

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{1}{2}\nabla n^2(\boldsymbol{r}). \qquad (2.19b)$$

Let α, β and γ are the angles which the ray makes with x, y, and $z-$axis respectively, then:

$$\boldsymbol{r} = s\cos\alpha\ \hat{i} + s\cos\beta\ \hat{j} + s\cos\gamma\ \hat{k}$$

therefore:

$$d\boldsymbol{r} = ds\ \cos\alpha\ \hat{i} + ds\ \cos\beta\ \hat{j} + ds\ \cos\gamma\ \hat{k} \qquad (2.20a)$$

by using Eq.(2.18) , we have:

$$\frac{d\boldsymbol{r}}{dt} = n\cos\alpha\ \hat{i} + n\cos\beta\ \hat{j} + n\cos\beta\ \hat{k}. \qquad (2.20b)$$

Define a unit optical ray vector, $\boldsymbol{T}$, as:

$$\boldsymbol{T} \equiv \frac{d\boldsymbol{r}}{dt} \qquad , |\boldsymbol{T}| = 1 \tag{2.21}$$

using Eq.(2.18) the optical ray vectors can be expressed as:

$$\boldsymbol{T} \equiv \frac{d\boldsymbol{r}}{dt} \equiv n\frac{d\boldsymbol{r}}{ds} \equiv n\frac{dx}{ds}\hat{i} + n\frac{dy}{ds}\hat{j} + n\frac{dz}{ds}\hat{k} \tag{2.22}$$

and by comparing Eq.(2.20b) with Eq.(2.22), the optical ray vector can be expressed as:

$$\boldsymbol{T} \equiv n\, cos\alpha\, \hat{i} + n\, cos\beta\, \hat{j} + n\, cos\gamma\, \hat{k} \tag{2.23}$$

Let $\boldsymbol{T}_1$ and $\boldsymbol{T}_0$ are the optical ray vectors after and before refraction at the optical surface, respectively; $\boldsymbol{N}$ is the unit vector normal to the optical surface at the point of the refraction as in Fig.(2.7), the vectorial form of Snell's law(18) (see appendix A) can be written as:

$$\boldsymbol{T}_1 = \boldsymbol{T}_0 + \Omega \boldsymbol{N}$$

where:

$$\Omega = \sqrt{n_1^2 - n_0^2 + \upsilon^2} - \upsilon$$

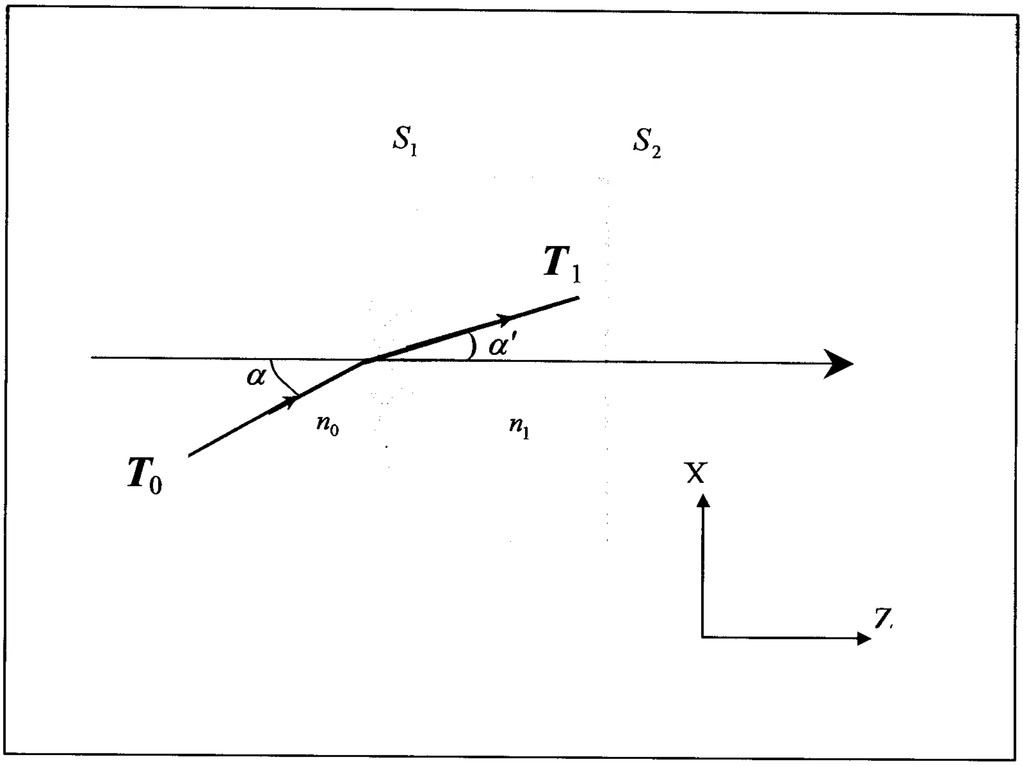


Fig.(2.7): Symbols used in ray tracing equation for axial GRIN.

where υ is the scalar product of $\boldsymbol{N}$ and $\boldsymbol{T}_0$, and n_1, n_0 are the refractive indices of the two media (after and before the optical surface at which the ray is refracted).

In a matrix form, ray position can be written as:

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2.24}$$

and the optical direction cosine yields:

$$\boldsymbol{T} = \begin{pmatrix} T_x \\ T_y \\ T_z \end{pmatrix} = n \begin{pmatrix} dx/ds \\ dy/ds \\ dz/ds \end{pmatrix} \tag{2.25}$$

where, the gradient of refractive index gives:

$$\boldsymbol{D} = \frac{1}{2} \begin{pmatrix} \partial n^2 / \partial x \\ \partial n^2 / \partial y \\ \partial n^2 / \partial z \end{pmatrix}. \tag{2.26}$$

To trace the ray through a medium, we will consider that the starting point is $(\boldsymbol{r}_0, \boldsymbol{T}_0)$, and Δt is the incremental value of t or the extrapolation distance. Then by using the Runge-Kutta algorithm(26,29) ,(for more detail about this algorithm(24) see

appendix B), we can express the successive position vector and the optical direction cosine of the ray $(\boldsymbol{r}_1, \boldsymbol{T}_1)$ as,

$$\boldsymbol{r}_1 = \boldsymbol{r}_0 + \Delta t\{\boldsymbol{T}_0 + \frac{1}{6}[\Delta t\boldsymbol{D}(\boldsymbol{r}_0) + 2\Delta t\boldsymbol{D}(\boldsymbol{r}_0 + \frac{1}{2}\Delta t\boldsymbol{T}_0 + \frac{1}{8}\Delta t^2\boldsymbol{D}(\boldsymbol{r}_0))]\} \tag{2.27a}$$

$$\boldsymbol{T}_1 = \boldsymbol{T}_0 + \frac{1}{6}\{\Delta t\boldsymbol{D}(\boldsymbol{r}_0) + 4[\Delta tD(\boldsymbol{r}_0 + \frac{1}{2}\Delta t\boldsymbol{T}_0 + \frac{1}{8}\Delta t^2\boldsymbol{D}(\boldsymbol{r}_0)]$$

$$+ \Delta t\boldsymbol{D}[\boldsymbol{r}_0 + \Delta t\boldsymbol{T}_0 + \frac{1}{2}\Delta t(\Delta t\boldsymbol{D}(\boldsymbol{r}_0 + \frac{1}{2}\Delta t\boldsymbol{T}_0 + \frac{1}{8}\Delta t^2\boldsymbol{D}(\boldsymbol{r}_0)))]\} \tag{2.27b}$$

To simplify Eqs.(2.27a-b) define the following notations as:

$$\mathrm{A}_0 = \Delta t\boldsymbol{D}(\boldsymbol{r}_0)$$

$$\mathrm{B}_0 = \Delta t\boldsymbol{D}[\boldsymbol{r}_0 + \frac{1}{2}\Delta t\boldsymbol{T}_0 + \frac{1}{8}\Delta t^2\boldsymbol{D}(\boldsymbol{r}_0)] = \Delta t\boldsymbol{D}[\boldsymbol{r}_0 + \frac{1}{2}\Delta t\boldsymbol{T}_0 + \frac{1}{8}\Delta t\mathrm{A}_0]$$

$$C_0 = \Delta t\boldsymbol{D}\{\boldsymbol{r}_0 + \Delta t\boldsymbol{T}_0 + \frac{1}{2}\Delta t[\Delta t\boldsymbol{D}[\boldsymbol{r}_0 + \frac{1}{2}\Delta t\boldsymbol{T}_0 + \frac{1}{8}\Delta t^2\boldsymbol{D}(\boldsymbol{r}_0)]]\}$$

$$= \Delta t\boldsymbol{D}[\boldsymbol{r}_0 + \Delta t\boldsymbol{T}_0 + \frac{1}{2}\Delta tB_0]$$

Therefore Eq.(2.27a) and Eq.(2.27b) read:

$$\boldsymbol{r}_1 = \boldsymbol{r}_0 + \Delta t[\boldsymbol{T}_0 + \frac{1}{6}(\mathrm{A}_0 + 2\mathrm{B}_0)], \tag{2.28a}$$

$$\boldsymbol{T}_1 = \boldsymbol{T}_0 + \frac{1}{6}(\mathrm{A}_0 + 4\mathrm{B}_0 + \mathrm{C}_0) \qquad (2.28b)$$

Eqs.(2.28a-b) can be generalized as:

$$\boldsymbol{r}_{j+1} = \boldsymbol{r}_j + \Delta t[\boldsymbol{T}_j + \frac{1}{6}(\mathrm{A}_j + 2\mathrm{B}_j)], \qquad (2.29a)$$

$$\boldsymbol{T}_{j+1} = \boldsymbol{T}_j + \frac{1}{6}(\mathrm{A}_j + 4\mathrm{B}_j + \mathrm{C}_j) \qquad (2.29b)$$

where $\boldsymbol{r}_j$ and $\boldsymbol{r}_{j+1}$, represent the position vectors of the point and its successive respectively, $\boldsymbol{T}_j$ and $\boldsymbol{T}_{j+1}$, represent the optical direction of the ray and its successive respectively. The matrices A_j,B_j, and C_j are defind as:

$$\mathrm{A}_j = \Delta t\boldsymbol{D}(\boldsymbol{r}_j)$$

$$\mathrm{B}_j = \Delta t\boldsymbol{D}[\boldsymbol{r}_j + \frac{1}{2}\Delta t\boldsymbol{T}_j + \frac{1}{8}\Delta t\mathrm{A}_j] \qquad (2.30)$$

$$\mathrm{C}_j = \Delta t\boldsymbol{D}[\boldsymbol{r}_j + \Delta t\boldsymbol{T}_j + \frac{1}{2}\Delta t\mathrm{B}_j]$$

In next section we will consider a special case of an axial and a radial GRIN , then we will give the solution of the ray equation in these media.

2.5 *Special cases:*

2.5.1 *The algebraic solution of a ray equation for a linear axial GRIN medium:*

Consider an optical system which lies in XZ-plane, located at air and containinng a plano-convex lens with a linear axial GRIN medium, as in Fig.(2.7). In this case we will consider the first two terms in Eq.(2.16a) :

$$n(z) = N_{00} + N_{01} z \tag{2.31}$$

where $N_{01} \lesseqgtr 0$

by taking the partial derivative of Eq.(2.31) with respect to x, y and z we will have:

$$\frac{\partial n}{\partial x} = 0, \quad \frac{\partial n}{\partial y} = 0 \quad \text{and} \quad \frac{\partial n}{\partial z} = N_{01},$$

then Eq.(2.12) can be written as:

$$\frac{d^2 x}{dt^2} = 0, \tag{2.32a}$$

$$\frac{d^2 y}{dt^2} = 0, \tag{2.32b}$$

$$\frac{d^2z}{dt^2} = N_{01}(N_{00} + N_{01}z) = N_{01}N_{00} + N_{01}^2 z. \quad (2.32c)$$

For a tangential ray, which lies in XZ-plane , $y = 0$, then Eqs.(2.32) have the following solutions:

$$x(t) = a_1 t + b_1 \quad (2.33a)$$

$$y(t) = a_2 t + b_2 \quad (2.33b)$$

$$z(t) = (c_1 e^{N_{01}t} + c_2 e^{-N_{01}t}) - \frac{N_{00}}{N_{01}} \quad (2.33c)$$

using the boundary conditions the all constants in Eqs.(2.33) will be determind. For this apply the first condition, $x(0) = 0$,into Eq.(2.33a), then:

$$b_1 = 0 \quad (2.34a)$$

and by applying the second condition, the derivative of Eq.(2.33a) with respect to t at $t = 0$, then:

$$[\frac{dx(t)}{dt}]_{t=0} = a_1. \quad (2.34b)$$

The optical direction cosine along x-direction is given as, (see Fig. 2.7):

$$[\frac{dx(t)}{dt}]_{t=0} = N_{01} \sin\alpha' \quad (2.35)$$

the comparison between Eq.(2.34a) and Eq.(2.35) gives:

$$a_1 = N_{01}\, sin\, \alpha' . \qquad (2.36a)$$

Applying Snell's law at the point of refraction (the vertex of the 1st surface, S_1) gives:

$$sin\, \alpha = N_{01} sin\, \alpha' \qquad (2.36b)$$

where (α, α') are the incident angle and the refracted angle of the ray at the vertex of the first surface,(see Fig.2.7).

By comparing Eq.(2.36a) and Eq.(2.36b) we have :

$$a_1 = \sin\alpha , \qquad (2.36c)$$

by substituting from Eq.(2.36c) and Eq.(2.34a) into Eq.(2.33a) we have:

$$x(t) = t\, \sin\alpha . \qquad (2.37)$$

Similarly, one can find that $a_2 = b_2 = 0$, therefore:

$y(t)=0$.

c_1 and c_2 can be determined as following:

Referring to Eq.(2.33c), $(z=0)_{t=0}$, then:

$$c_1 + c_2 = \frac{N_{00}}{N_{01}} \qquad (2.38a)$$

and $\quad (\frac{dz}{dt})_{t=0} = N_{01}(c_1 - c_2)$ (2.38b)

but from Fig.(2.7), the optical direction cosine along z-axis is given as:

$$(\frac{dz}{dt})_{t=0} = N_{01}\cos\alpha' = \sqrt{N_{01}^2 - \sin^2\alpha}$$

then Eq.(2.38b), becomes:

$$c_1 - c_2 = \frac{1}{N_{01}}\sqrt{N_{01}^2 - \sin^2\alpha} \tag{2.38c}$$

by solving Eqs.(2.40a,c) immediately,then:

$$c_1 = \frac{1}{2N_{01}}\left[N_{00} + \sqrt{N_{01}^2 - \sin^2\alpha}\right] \tag{2.39}$$

$$c_2 = \frac{1}{2N_{01}}\left[N_{00} - \sqrt{N_{01}^2 - \sin^2\alpha}\right] \tag{2.40}$$

Therefore, by using Eqs.(2.39) and (2.40) , Eq.(2.33c) becomes:

$$z(t) = \frac{N_{00}}{N_{01}}\left[\cosh(N_{01}t) + \sqrt{\frac{N_{01}^2 - \sin^2\alpha}{N_{00}^2}}\sinh(N_{01}t) - 1\right] \tag{2.41}$$

since t is a parameter , one can evaluate z as a function of x as follows:

from Eq.(2.37) we have :

$$t = \frac{x}{sin\alpha}$$

therefore Eq.(2.41) can be written as:

$$z(x) = \frac{N_{00}}{N_{01}} \left[cosh(\frac{N_{01}}{sin\alpha} x) + \sqrt{1 - \frac{sin^2\alpha}{N_{00}^2}} sinh(\frac{N_{01}}{sin\alpha} x) - 1 \right] \quad (2.42)$$

For $\alpha = 0$, Eq.(2.41) reads:

$$z(t) = \frac{N_{00}}{N_{01}} \left[cosh(N_{01}t) + sinh(N_{01}t) - 1 \right] \quad (2.43)$$

expressing the *sinh* and *cosh* functions in its exponential forms in Eq.(2.43) gives:

$$z(t) = \frac{N_{00}}{N_{01}} \left[e^{N_{01}t} - 1 \right] \quad (2.44)$$

consider a weak axial GRIN where $N_{01}t \ll 0$, then:

$$e^{N_{01}t} \approx 1 + N_{01}t$$

therefore, Eq.(2.44) becomes:

$$z(t) \approx N_{00}t. \quad (2.45)$$

This equation indicates that the path of the ray will be straight line in this medium.

2.5.2 *The algebraic solution of a ray equation for a radial GRIN medium:*

Consider a medium that has the following gradient index:

$$n^2(r) = N_{00}^2 \pm N_{10}^2 r^2 \qquad (2.46)$$

where: $N_{10} > 0$,

For this medium it is best to adopt Cartesian coordinates and to convert Eq.(2.12) to derivatives with respect to z instead of s[16].

In this medium, $\frac{\partial n}{\partial z} = 0$, so that, in any radial gradient, Eq.(2.19a) can be reduced to the form:

$$\ddot{\boldsymbol{r}} = \frac{n(\boldsymbol{r})\nabla n(\boldsymbol{r})}{T_z^2} \qquad (2.47)$$

where T_z is the optical direction cosine in the direction of $z-$axis.

Eq.(2.47) is often inconvenient, for its component equations are two second-order differential equations that are coupled, both right-hand sides being functions of x, y and r. However, by considering the negative sign in Eq.(2.46), we have :

$$n(\boldsymbol{r})\nabla n(\boldsymbol{r}) = \frac{1}{2}\nabla n^2(\boldsymbol{r}) = -N_{10}^2\boldsymbol{r} \tag{2.48}$$

Then Eq.(2.49) reduces to

$$\ddot{\boldsymbol{r}} + \frac{\boldsymbol{r}N_{10}^2}{T_{0z}^2} = 0 \tag{2.49}$$

if we modify the $z-$coordinate by putting

$$\eta = \frac{(Z - z_0)}{T_{0z}} N_{10} \tag{2.50}$$

where the coordinates Z is the parameter that specify the axial distance of the ray position inside the GRIN medium, then we have:

$$\frac{d^2x}{d\eta^2} + x = 0 \tag{2.51a}$$

$$\frac{d^2y}{d\eta^2} + y = 0 \tag{2.51b}$$

The solution of these equations is given by[(16)]:

$$x = A_1 \cos\eta + B_1 \sin\eta \tag{2.52a}$$

$$y = A_2 \cos\eta + B_2 \sin\eta \tag{2.52b}$$

As in Ref.(16) using the initial coordinates and the initial direction cosine of the incident ray at first surface,we can determine the constants A_1, B_1 , A_2 and B_2 as the following:

$$A_1 = x_0, \qquad B_1 = \frac{T_{0x}}{N_{10}},$$

$$A_2 = y_0, \qquad B_1 = \frac{T_{0x}}{N_{10}}.$$

Considering the positive sign in Eq.(2.46), the differential equations reduce to:

$$\frac{d^2x}{d\eta^2} - x = 0 \tag{2.53a}$$

$$\frac{d^2y}{d\eta^2} - y = 0 \tag{2.53b}$$

the solutions of these equations is given as:

$$x = x_0 \cosh\eta + \frac{T_{0x}}{N_{10}} \sinh\eta \tag{2.54a}$$

$$y = y_0 \cosh\eta + \frac{T_{0y}}{N_{10}} \sinh\eta \tag{2.54b}$$

Chapter 3

PROFILES OF A RAY PATH IN GRIN

The ray trajectory through a nonhomogeneous medium, where the refractive index is a function of position, satisfies Ray-Equation (RE). A ray-tracing program, which specifies the trajectory of ray through the GRIN lens, is written. This program gives the numerical solution of RE. Therefore the effect of GRIN on the ray trajectory are studied.

In next section, the mathematical procedures of the ray tracing [4,23] in an optical system will be presented.

3.1 *Ray Tracing procedures*:

Consider an optical system ,(see Fig.3.1), then from a fixed object point located at (x_0, y_0, z_0) an initial ray is chosen. Consider a unit vector along this ray, A_0, lies in xz-plane, then:

$$A_0 = sina\,\hat{i} + cos\alpha\,\hat{k} \tag{3.1}$$

where α is the angle between this ray and z-axis.

The point of intersection of this ray with first surface, (x_1, y_1, z_1) ,can be found as following:

From Fig.(3.1), we have:

$$tana = \frac{x_1 - x_0}{z_1 - z_0} \tag{3.2}$$

therefore:

$$x_1 = (z_1 - z_0)tana + x_0 \tag{3.3}$$

$$y_1 = y_0 \tag{3.4}$$

z_1 is determined from surface equation as follows:

$$h_1^2 + (R - z_1)^2 = R^2 \tag{3.5}$$

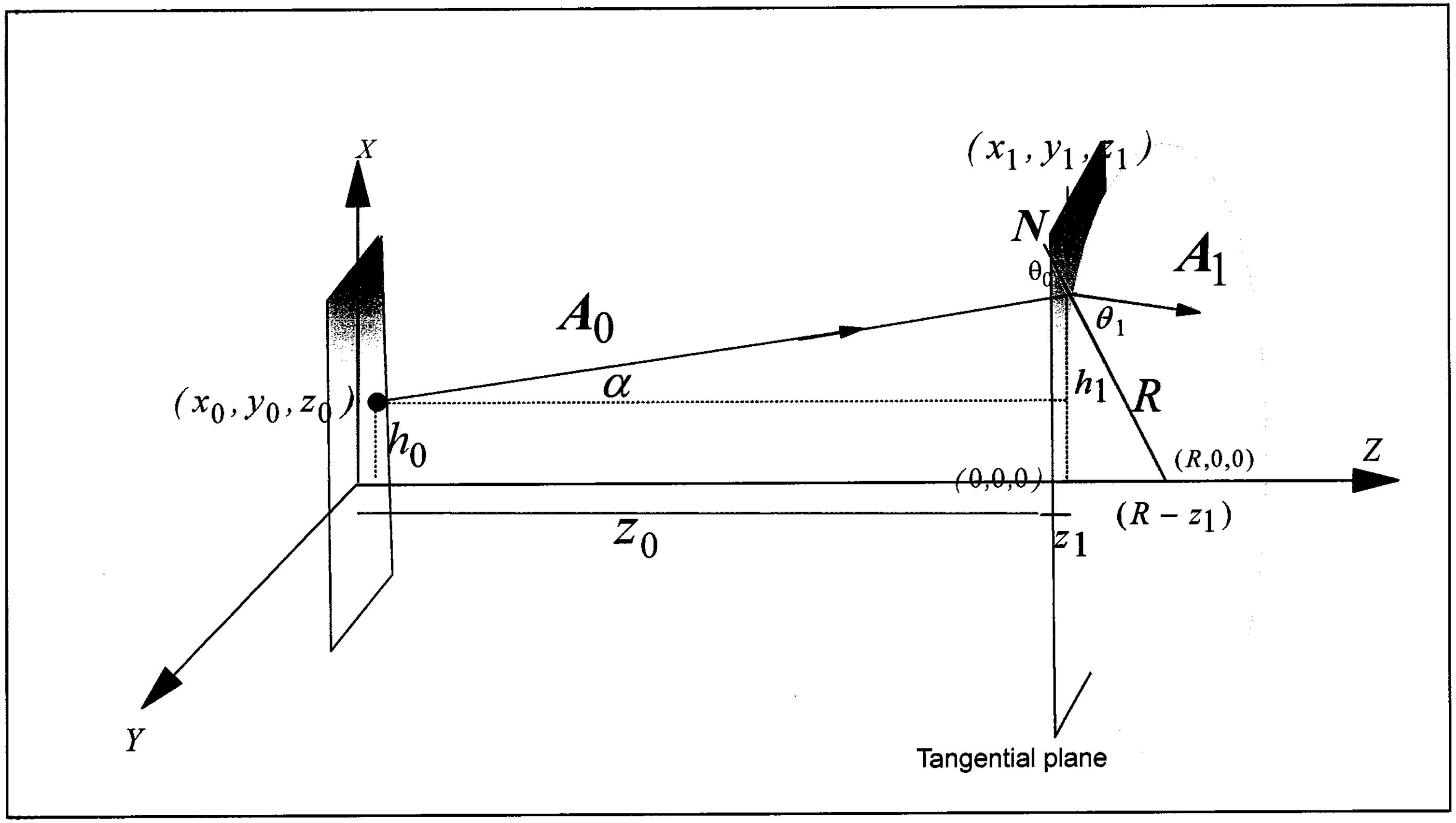


Fig.(3.1):An optical system used for ray tracing.

since:

$$h_1^2 = x_1^2 + y_1^2$$

then, Eq.(3.5) reads:

$$(x_1^2 + y_1^2) - 2Rz_1 + z_1^2 = 0 \tag{3.6}$$

substituting Eq.(3.3) into Eq.(3.6) gives

$$(1 + tan^2\alpha)z_1^2 + (-2z_0 tan^2\alpha + 2x_0 tan\alpha - 2R)z_1$$

$$+ x_0^2 + y_0^2 + 2x_0 z_0 tan\alpha + z_0^2 tan^2\alpha = 0 \tag{3.7}$$

then, the solution of Eq.(3.7) for z_1 is given as:

$$z_1 = \frac{-q_2 \pm \sqrt{q_2^2 - 4q_1 q_3}}{2q_1} \tag{3.8}$$

where:

$$q_1 = 1 + tan^2\alpha,$$

$$q_2 = -2z_0 tan^2\alpha + 2x_0 tan\alpha - 2R,$$

$$q_3 = x_0^2 + y_0^2 + 2x_0 z_0 tan\alpha + z_0^2 tan^2\alpha.$$

The desired solution of Eq.(3.8), must satisfy the condition $z_1 > 0$, then one can find x_1 from Eq.(3.3).

The unit normal vector, $\boldsymbol{N}$, at the first refracting surface intersection point can be calculated from surface equation written in polar coordinates:

$$\boldsymbol{X}(s,\varphi) = s\cos\varphi\,\hat{i} + s\sin\varphi\,\hat{j} + z(s)\hat{k} \qquad (3.9a)$$

where:

$\boldsymbol{X}(s,\varphi)$, is a vector function of two parameters of polar coordinates (s,φ). φ is a polar coordinate angle measured from the positive x-axis. $s^2 = x^2 + y^2$, and , $z(s) = cs^2 / 1 + \sqrt{1 - c^2 s^2}$

From the definition[18] of $\boldsymbol{N}$:

$$\boldsymbol{N} = \frac{\dfrac{\partial \boldsymbol{X}(s,\varphi)}{\partial \varphi} \times \dfrac{\partial \boldsymbol{X}(s,\varphi)}{\partial s}}{\left| \dfrac{\partial \boldsymbol{X}(s,\varphi)}{\partial \varphi} \times \dfrac{\partial \boldsymbol{X}(s,\varphi)}{\partial s} \right|} \qquad (3.9b)$$

where:

$$\frac{\partial \boldsymbol{X}}{\partial \varphi} = -s\sin\varphi\hat{i} + s\cos\phi\hat{j}$$

$$\frac{\partial \boldsymbol{X}}{\partial s} = -\cos\phi\hat{i} + \sin\varphi\hat{j} + \frac{\partial z(s)}{\partial s}$$

and,

$$\frac{\partial X}{\partial \varphi}\times\frac{\partial X}{\partial s}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -s\,sin\,\varphi & s\,cos\,\varphi & 0\\ cos\,\varphi & sin\,\varphi & \partial z(s)/\partial s\end{vmatrix}$$

$$=s\frac{\partial z(s)}{\partial s}cos\,\varphi\hat{i}+s\frac{\partial z(s)}{\partial s}sin\,\varphi\hat{j}-s\hat{k},$$

where, $\dfrac{\partial z(s)}{\partial s}=\sqrt{\dfrac{s^2c^2}{1-s^2c^2}}$,

then the unit normal vector is given as:

$$N=\frac{\dfrac{sc}{\sqrt{1-c^2s^2}}cos\,\varphi\hat{i}-\dfrac{c}{\sqrt{1-c^2s^2}}sin\,\varphi\hat{j}+\hat{k}}{\sqrt{1+\dfrac{s^2c^2}{1-s^2c^2}}}. \qquad (3.9c)$$

From Snell's law, the direction cosines of the incident and refracted rays are realated by,(see appendix A):

$$A_1=\gamma A_0+\Omega N \qquad (3.10)$$

where A_0 , A_1 are unit vectors along the incident and refracted ray and:

$$\gamma=\frac{n_0}{n_1}, \qquad (3.11a)$$

$$\Omega = \gamma \cos\theta_0 - \sqrt{1-(\gamma \sin\theta_0)^2} \qquad (3.11b)$$

After the ray is refracted at the first surface ,the position and the direction of this ray through the GRIN medium can be determined by using Eqs.(2.29a) and (2.29b).

By repeating application of Eqs.(3.8),(3.3),(2.29a) and (2.29b), one is able to trace rays through an optical system.

In next section we are going to get a profile of the ray in a GRIN media .

3.2 *Profiles of ray path in GRIN media*:

To know the profiles of ray path in GRIN medium, we consider a certain lens as a first step then we trace the ray inside this lens by designing a special program enable us to do so. In our study a plano-convex lens is used, this lens is specifide by the following parameter : The curvature of the anterior surface $(C_1 = 0.05mm^{-1})$, the curvature of the posterior surface $(C_2 = 0.0mm^{-1})$, and thickness $(d = 3mm)$, a diagram of this lens is shown in Fig.(3.2).

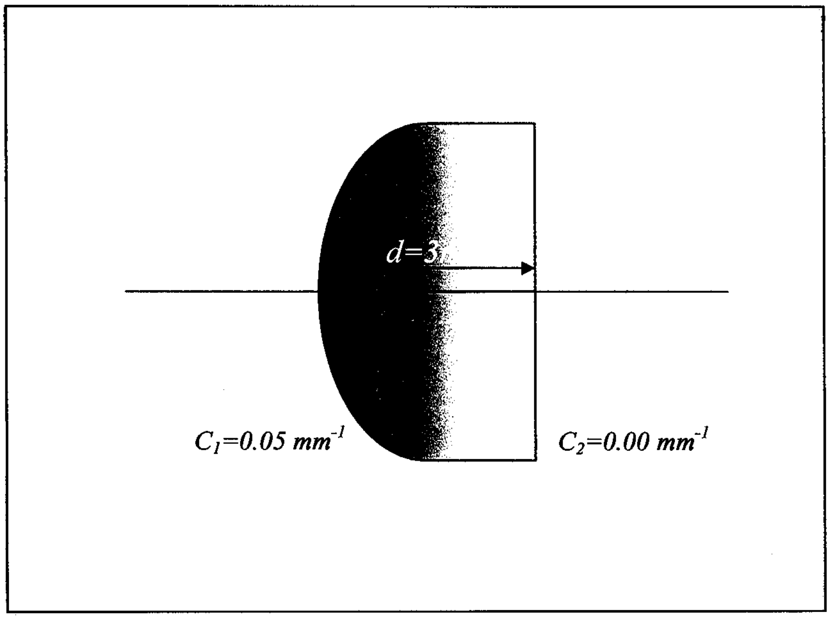


Fig.(3.2): Illustration of the lens layer.

Depending on the solution of the ray equations[26], and the shape of lens which given above, the ray trace algorithm is supplied with the angle of incidence. At different heights of the aperture, the ray is traced to obtain the final directions, and coordinates of the ray through the gradient lens. This algorithm is written in a Fortran program (see appendix C)

In next section, we will present a program which is used to trace the ray inside a linear axial GRIN. Then we will use this prgram to investigate the characteristic of the ray path in this medium.

3.2.1 *The profiles of the ray in a linear axial GRIN medium*:

In this case the refractive index can be written as:

$$n(z) = N_{00} + N_{01}z$$

By using the test lens which is given in Fig.(3.2) and by applying the ray tracing equations , the program which is given in appendix C is used to trace the ray inside this linear axial GRIN.

To study the effect of gradient parameter on a ray path inside this test lens, we will use firstly a lens with parameters $N_{00} = 1.76$ and $N_{01} = 0.0$,(homogeneous lens) . Then by running our program, the refractive index profiles of a homogeneous lens is shown in Fig.(3.3), here we can see that the refractive index profile is a straight line remains constant. The ray path which gives the ray height as a function of the axial distance for angle of incidence (5^0, 10^0, 15^0, and 20^0) through this lens is shown in Fig. (3.4). Clearly, this path is a stright line, which realized Eq.(2.15).

Now by entering a gradient parameter for a linear axial GRIN ,N_{01}, on this lens we can study the effect of gradient refractive index on ray path. The refractive index profile in this linear axial GRIN, (see Fig.3.5), can be studied by choosing a negative and a positive sign of N_{01}. We can see that for a negative sign, as the axial distance increases the refractive index decreases. While for a positive sign, as the axial distance increases the refractive index increases.

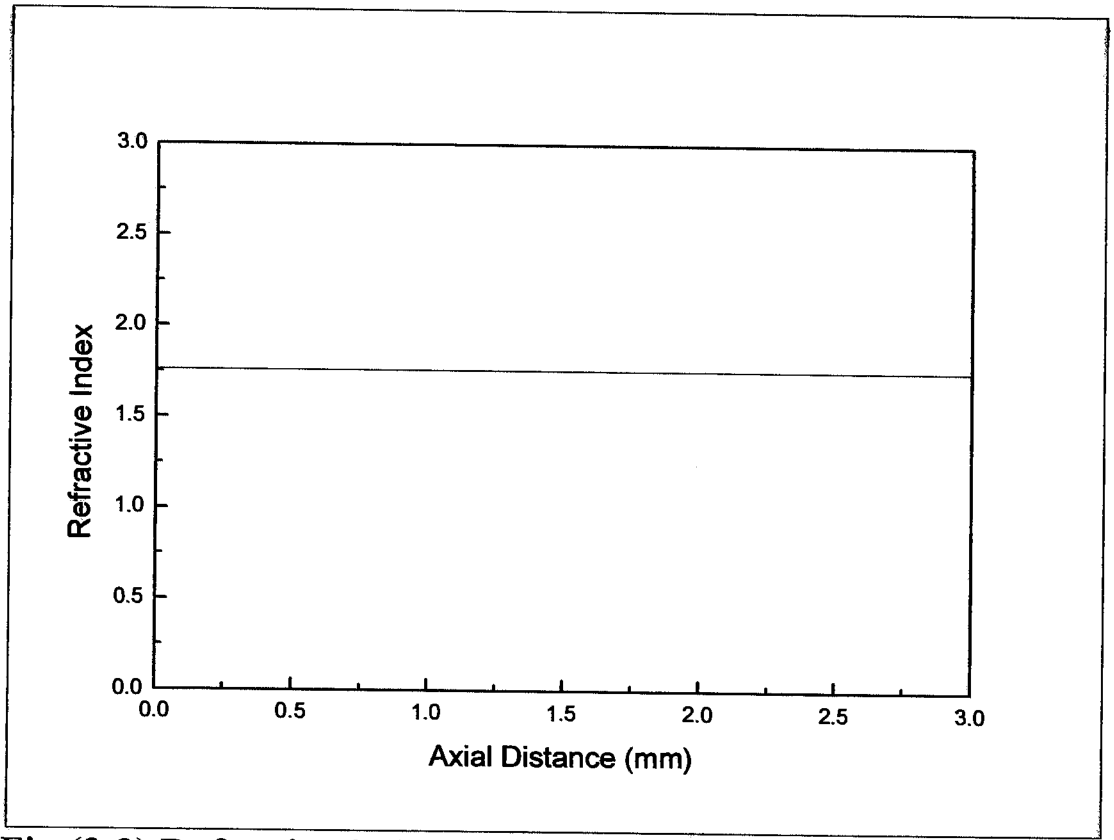


Fig.(3.3):Refractive index profile in a homogeneous lens.

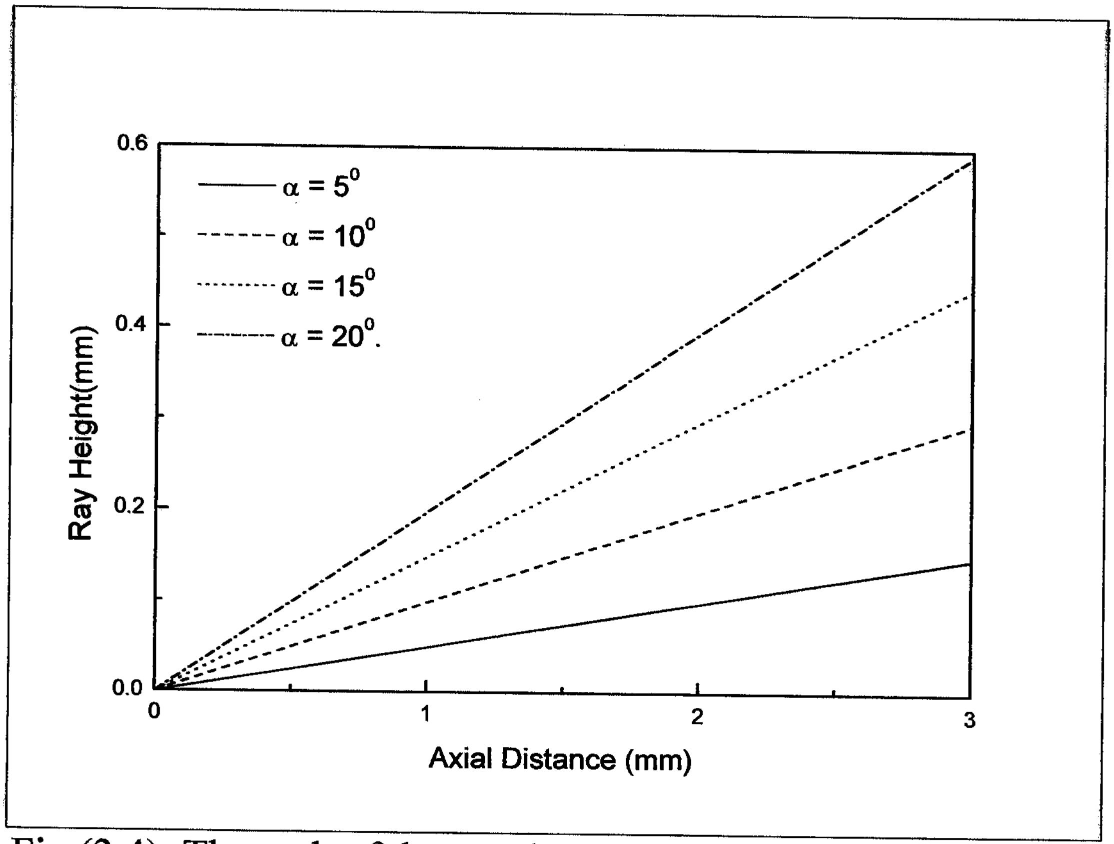


Fig.(3.4): The path of the ray through a homogeneous lens.

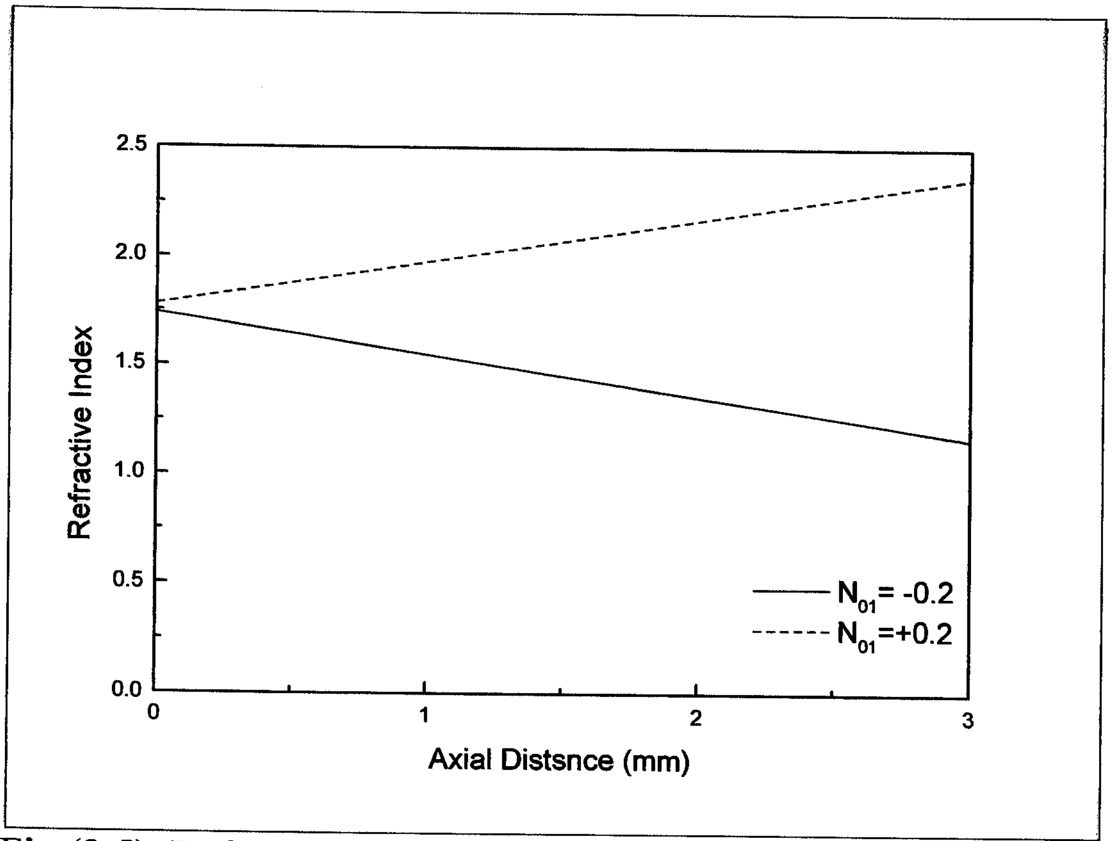


Fig.(3.5): Refractive index profile in a linear axial GRIN lens.

To study the ray path in this lens we consider three cases. In the first , we fix the value of α , the angle which the incident ray makes with optical axis, and vary the value of N_{01} . Secondly we fix the value of N_{01} while we vary the value of α. Finally the effect of gradient parameter's sign (negative and positive) at a certain value of α and N_{01} on ray path is graphed.

For the first case , we consider a different values of N_{01} such as: (-0.05, -0.1, -0.15, and -0.2), and $\alpha = 20^0$. After running our computer program, we graph the relation between the axial distance along the optical axis and the ray height (ray path), (see Fig.3.6). From this figure one can see that when $N_{01} = -0.05$,the ray path looks like a straight line (weak GRIN), while at $N_{01} = -0.1$, ray path takes a curve line more clearly. By taking $N_{01} = -0.15$ and $N_{01} = -0.2$, the slope of the curve increases with the axial distance.

For second case we consider a fixed value of gradient such as $N_{01} = -0.2$, then we consider 'α' as a varible value such as (5^0 ,10^0 ,15^0 and 20^0). For this case, the ray path is shown in

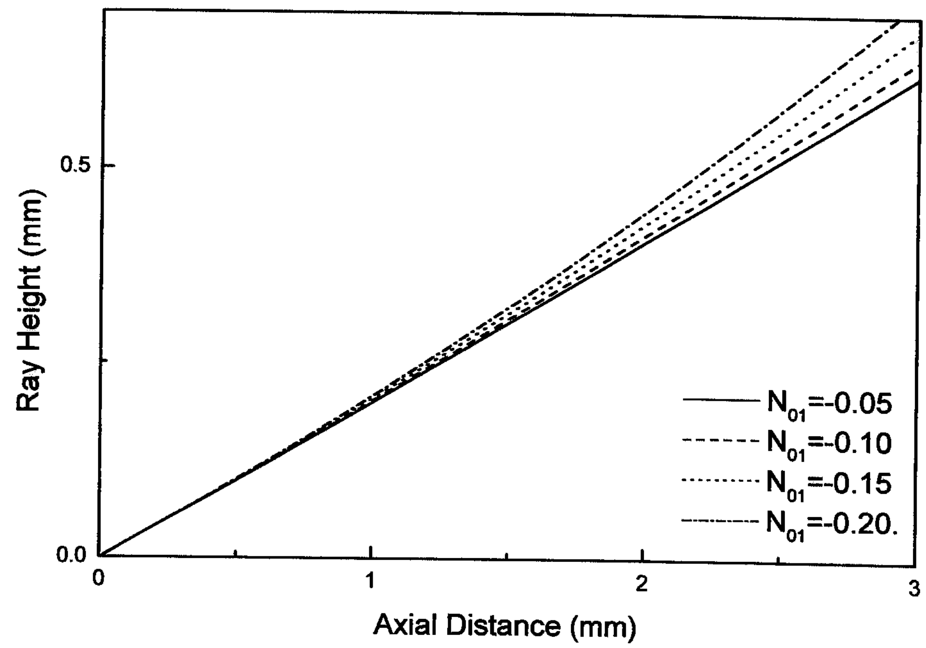


Fig.(3.6): The path of the ray inside a linear axial GRIN at $\alpha = 20^0$.

Fig.(3.7). From this figure one can see that if the ray makes a large angle with optical axis then its path will take a curve with a higher slope, while for a small angle, the ray path is a curve with a less slope.

To study the effect of a negative and a positive gradient parameters on the ray path through this lens. For $\alpha = 20^0$, we graph the ray path as shown in Fig.(3.8), for $N_{01} = +0.2$, the path takes a curve line more than the ray path for $N_{01} = -0.2$. This is explaind by Snell's law which implies that if light passes from a medium with lower index of refraction to one with a higher index refraction the light is bent tword the normal, and if the light passes from a higher to lower index of refraction the light bent away from the normal(10).

In next section we will discuss the profiles of the ray inside a quadratic axial GRIN medium.

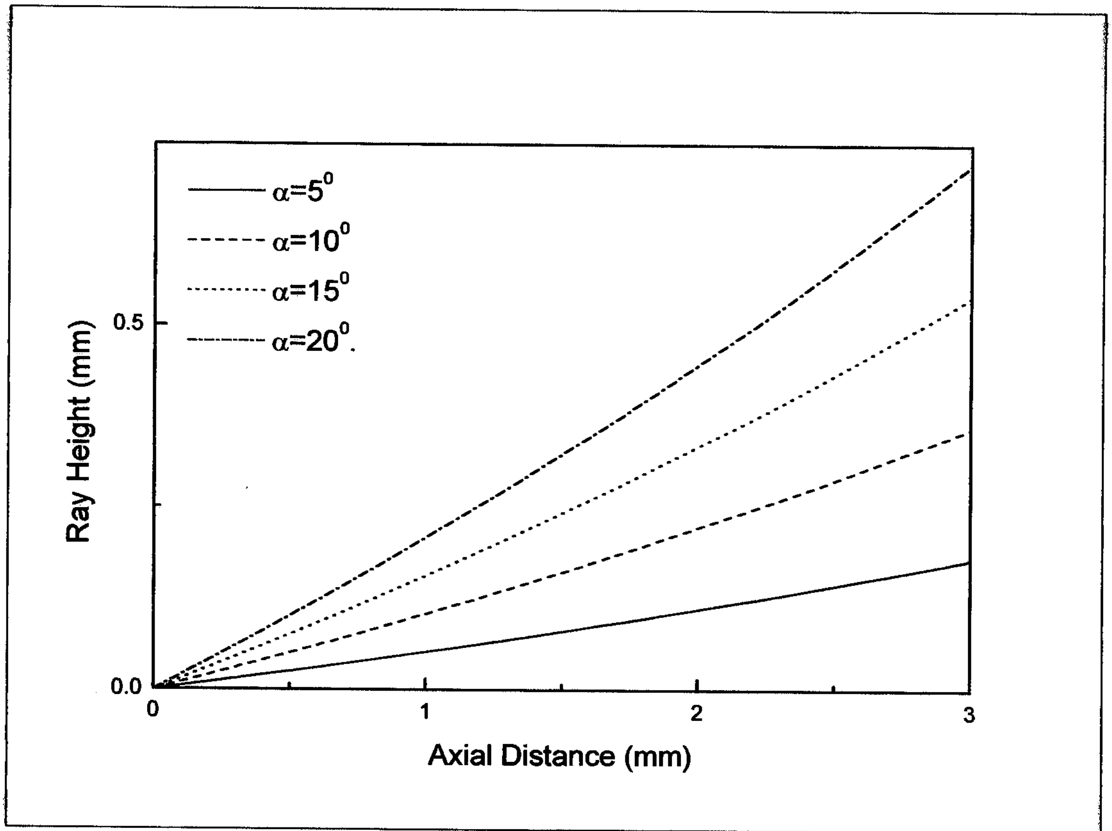


Fig.(3.7): The ray path in a linear axial GRIN at N_{01}= -0.2 and different incident angles.

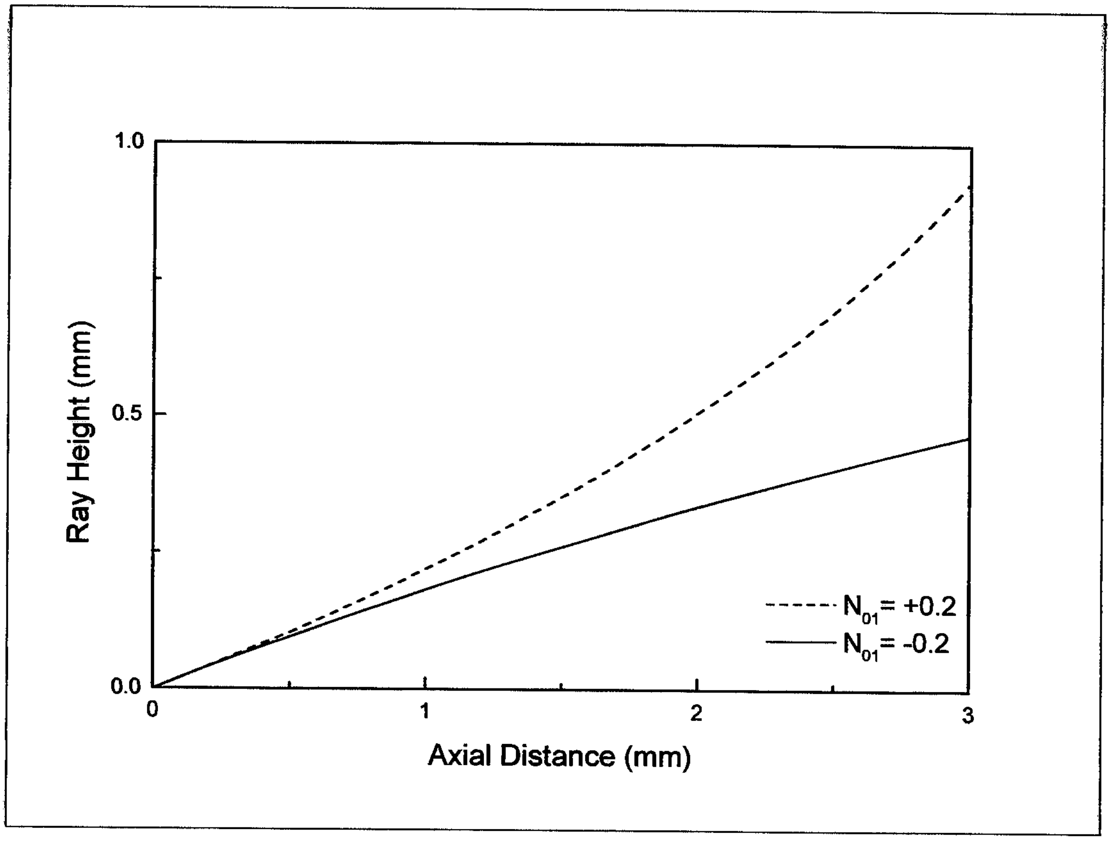


Fig.(3.8): The effect of a positive and a negative sign of GRIN parameter on the path of the ray through a linear axial GRIN lens.

3.2.2 *The profiles of the ray in a quadratic axial GRIN medium*:

The refractive index is expressed as:

$$n(z) = N_{00} + N_{01}z + N_{02}z^2$$

By using the program which is written for this case, (see appendix D), and using the same test lens parameters which we are used in section (3.2.1), the ray path through this lens can be determined. After running the program, the profiles of refractive index as a function of the GRIN parameters graphed (see Fig.3.9).These profiles are determined for many values of ($N_{01} \gtrless 0$) and ($N_{02} \gtrless 0$).

The profiles of ray path in a quadratic axial GRIN can be studied in many cases. In the first case, we fix the values of $N_{01} = -0.2$ and $\alpha = 20^0$, while the values of N_{02} are varible such as: ±0.14, ±0.096, ±0.052 and ±0.008. The ray paths are determined and graphed (see Fig.3.10). Referring to this figure, one can see that if N_{02} is negative then the ray takes a small

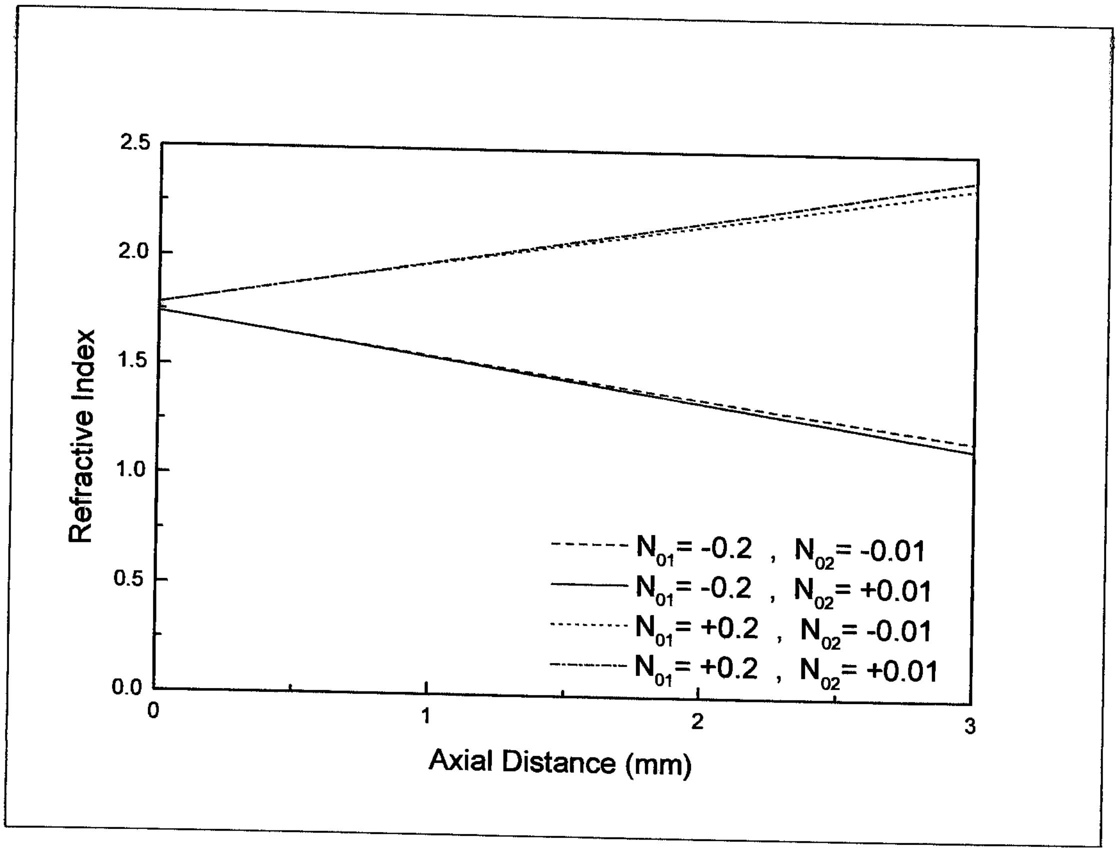


Fig.(3.9): Index profiles for a quadratic axial GRIN medium.

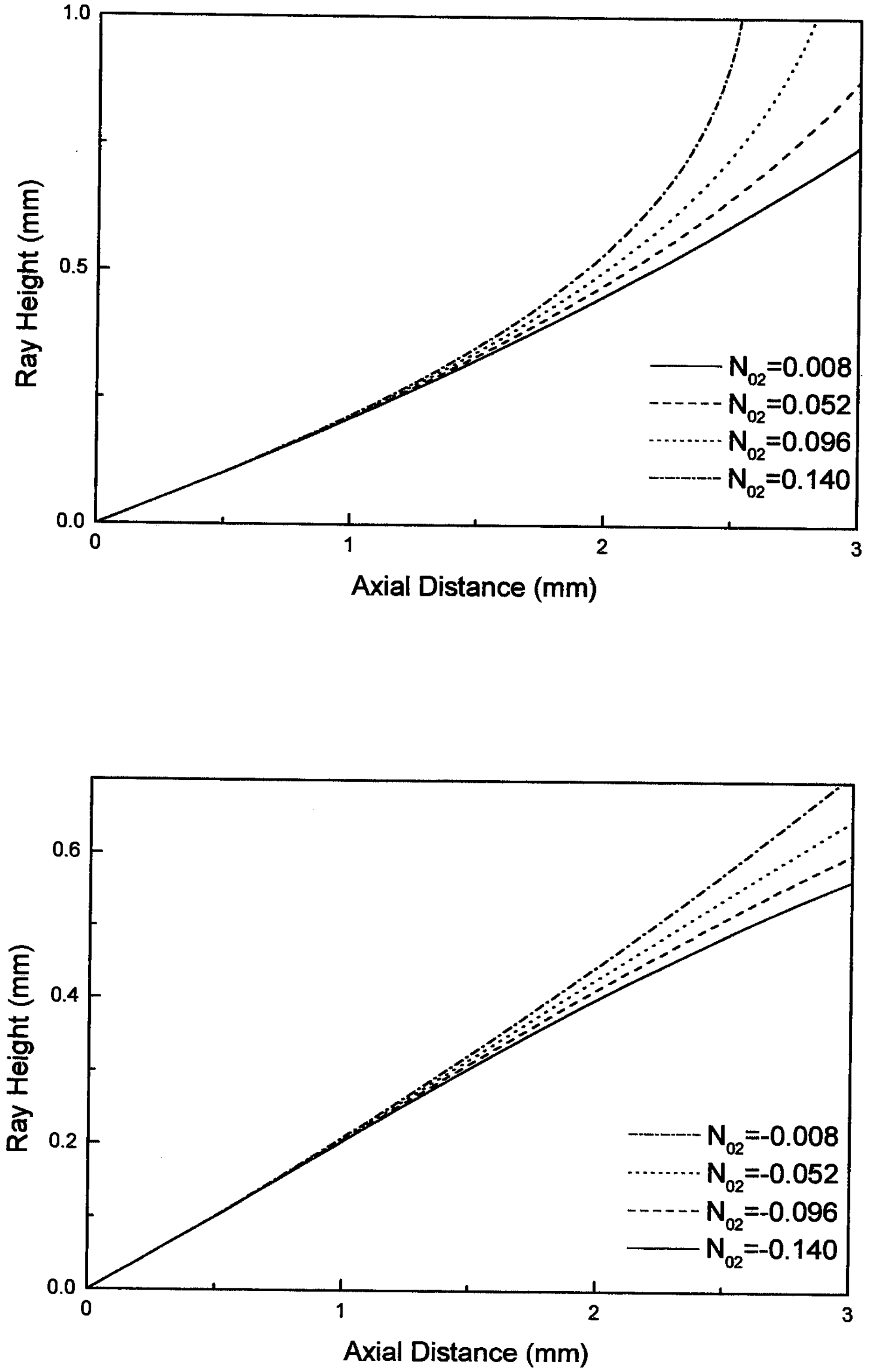


Fig.(3.10): The path of the ray through a quadratic axial GRIN lens at $\alpha = 20^0$ and N_{01}= -0.2.

slope through the lens , while if N_{02} is a positive then the ray takes a high slope.

In second studying, we fix the value $N_{01} = 0.2$ and $\alpha = 20^0$, while we vary the value of N_{02} such as : ±0.14, ±0.096, ±0.052 and ±0.008 . Then the rays paths are determined and graphed (see Fig.3.11). Referring to this figure, one can see the effect of N_{02}'s sign on the profiles of the ray path through the GRIN medium.

In third studying, we fix the value of $N_{01} = -0.2$ and $N_{02} = -0.01$, while α is a varied such as:(5^0,10^0 ,15^0 and 20^0). The effect of varying of the incident angle on ray path through the GRIN medium is investigated (see Fig.3.12). Referring to this figure, one can see that if the incident angles increases the ray paths becomes more curved.

To compare the ray path inside two lenses, the first one with a linear axial GRIN and the other one with a quadratic axial GRIN, we choose a certain values of N_{01}, N_{02} and α such as

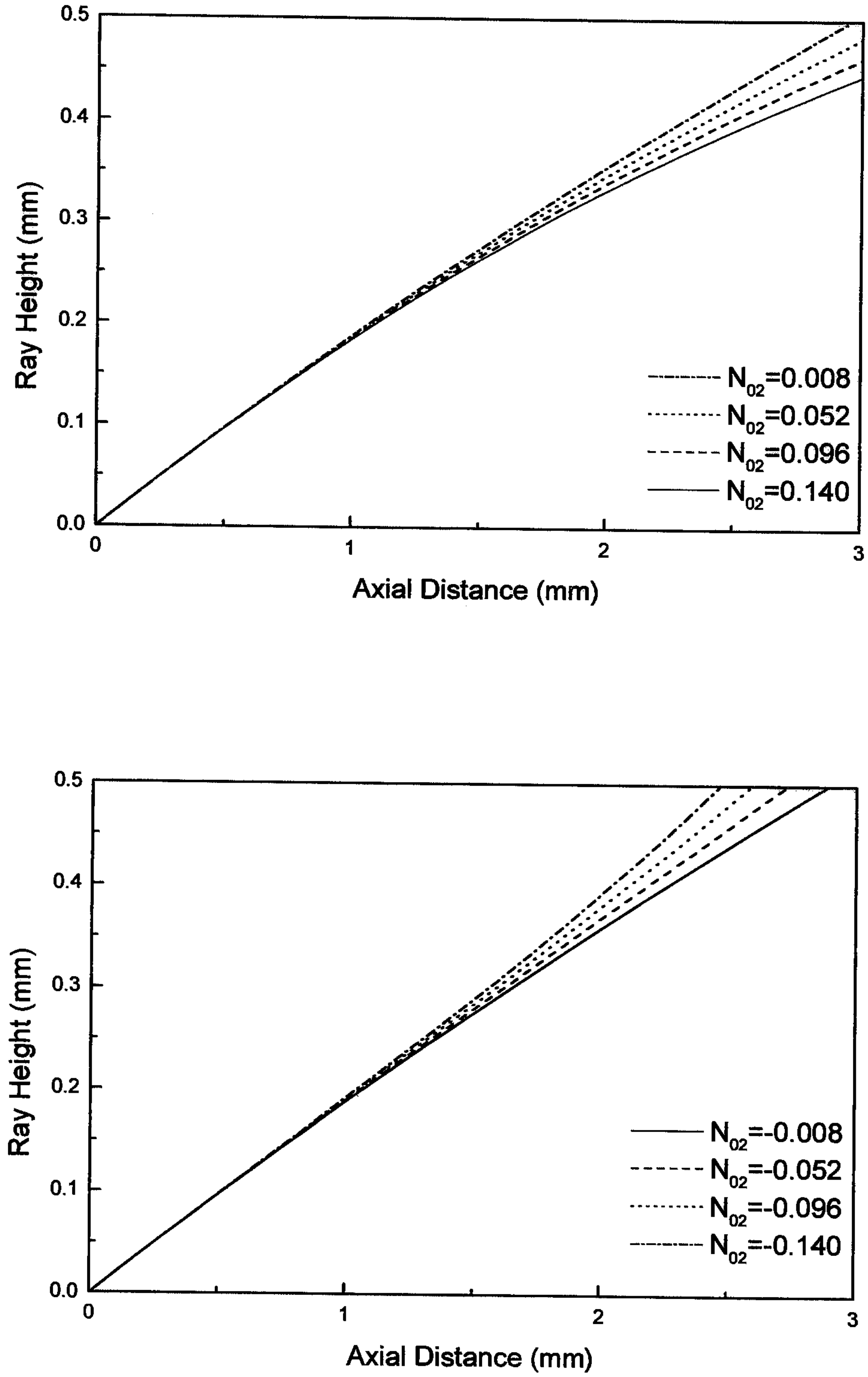


Fig.(3.11): The path of the ray through a quadratic axial GRIN lens at $\alpha = 20^0$ and N_{01}= +0.2.

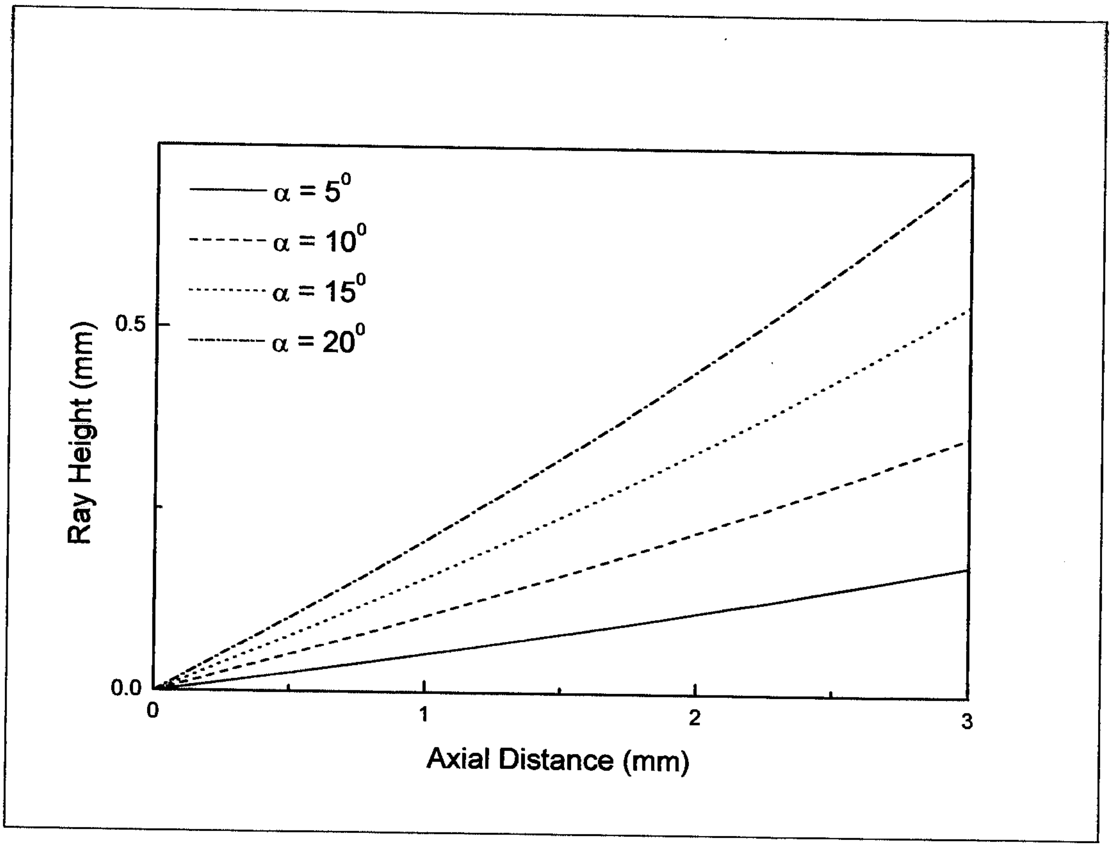


Fig.(3.12): The path of the ray through a quadratic axial GRIN lens at different angles and at $N_{01} = -0.2$ and $N_{02} = -0.01$.

($\alpha = 20^0, N_{01} = -0.2$ and $N_{02} = -0.14$). The study of these cases can be shown in Fig.(3.13) .

In the next section, we will consider a radial GRIN medium. The profiles of the ray inside this medium will be investigated.

3.2.3 *The profile of the ray inside radial GRIN medium*:

If we take the following refractive index:

$$n(r) = N_{00} + N_{10} r^2.$$

By using the same lens parameters which we used it in section 3.2.1, the FORTRAN program related to this kind of lens is given in appendix E. The profiles of refractive index for positive and negative values of N_{10}, can be shown in Fig.(3.14) The ray path through this lens can be studied in many ways . On one hand, we will fix the value of α, while the value of N_{10} is varied such as(-0.001, -0.01, -0.02, -0.03). On other hand, we take N_{10} as a constant value , while the value of α is varied such as (10^0, 5^0, and 3^0).

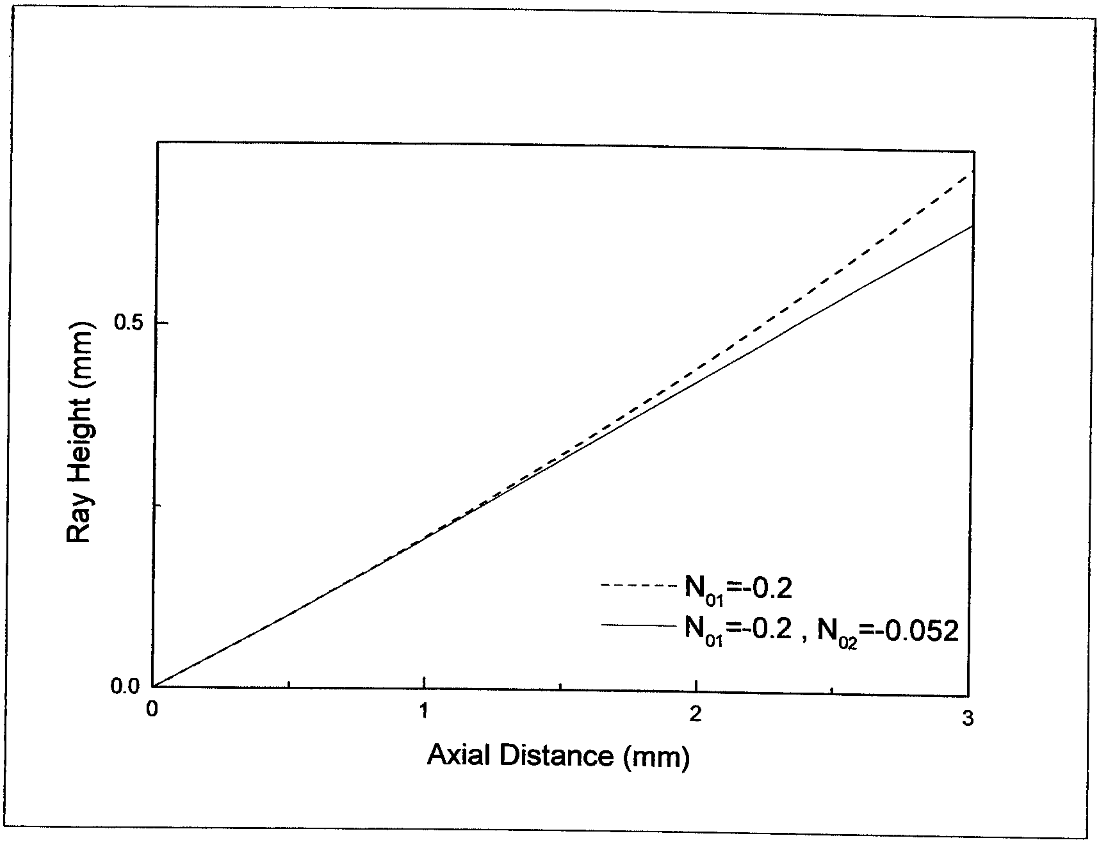


Fig.(3.13): The path of the ray inside a linear GRIN lens (----), and a quadratic axial GRIN lens (—).

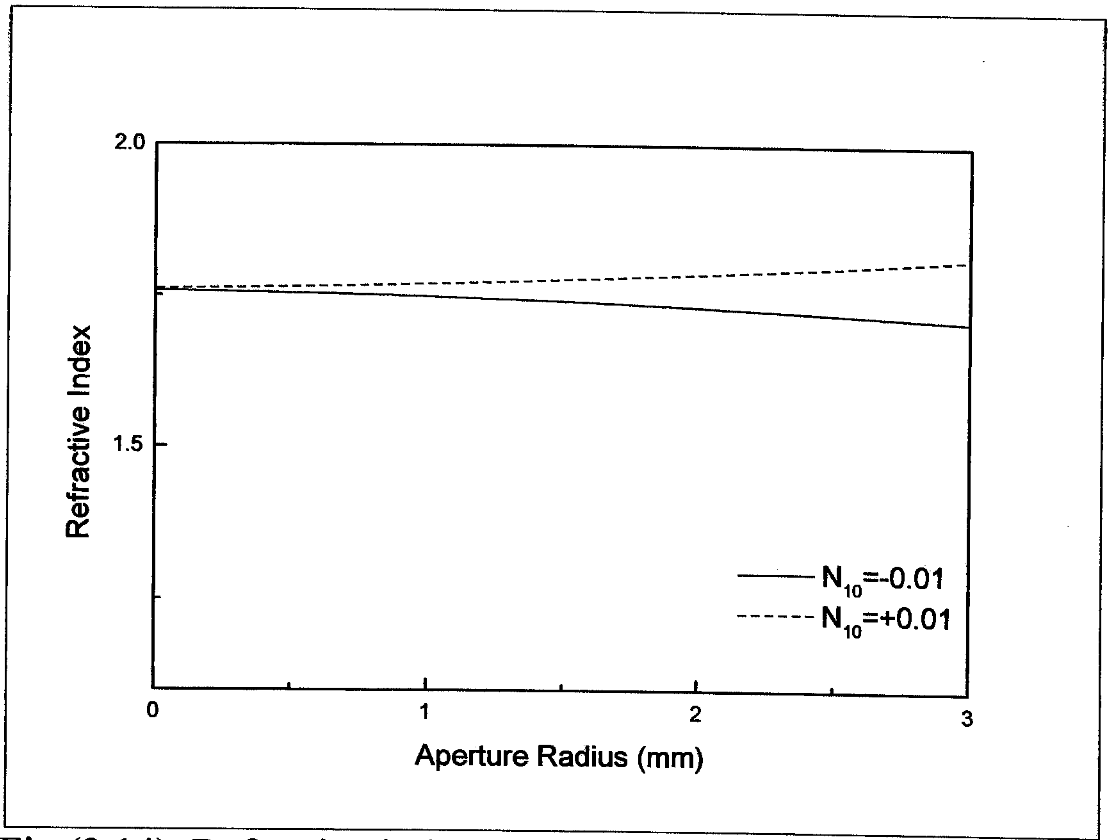


Fig.(3.14): Refractive index profiles in a radial GRIN lens.

The first way can be shown in Fig.3.15, from this figure we can see that for a large value of N_{10} , ray path becomes more curved .

The second way can be shown in Fig.3.16, from this figure we can see that as the value of α increases ray height increases and the ray path becomes more curved.

To study the effect of a positive and a negative sign of gradient parameter , N_{10}, on the ray path through this lens, we graphed the ray path for both values of N_{10} ,$\pm$ 0.03, (see Fig.3.17). This figure indicates that for a positive sign of N_{10}, the ray path has a concaved shape, while for negative value of N_{10}, the ray path has a convexed shape. This result satisfies Snell's law of refraction through lens.

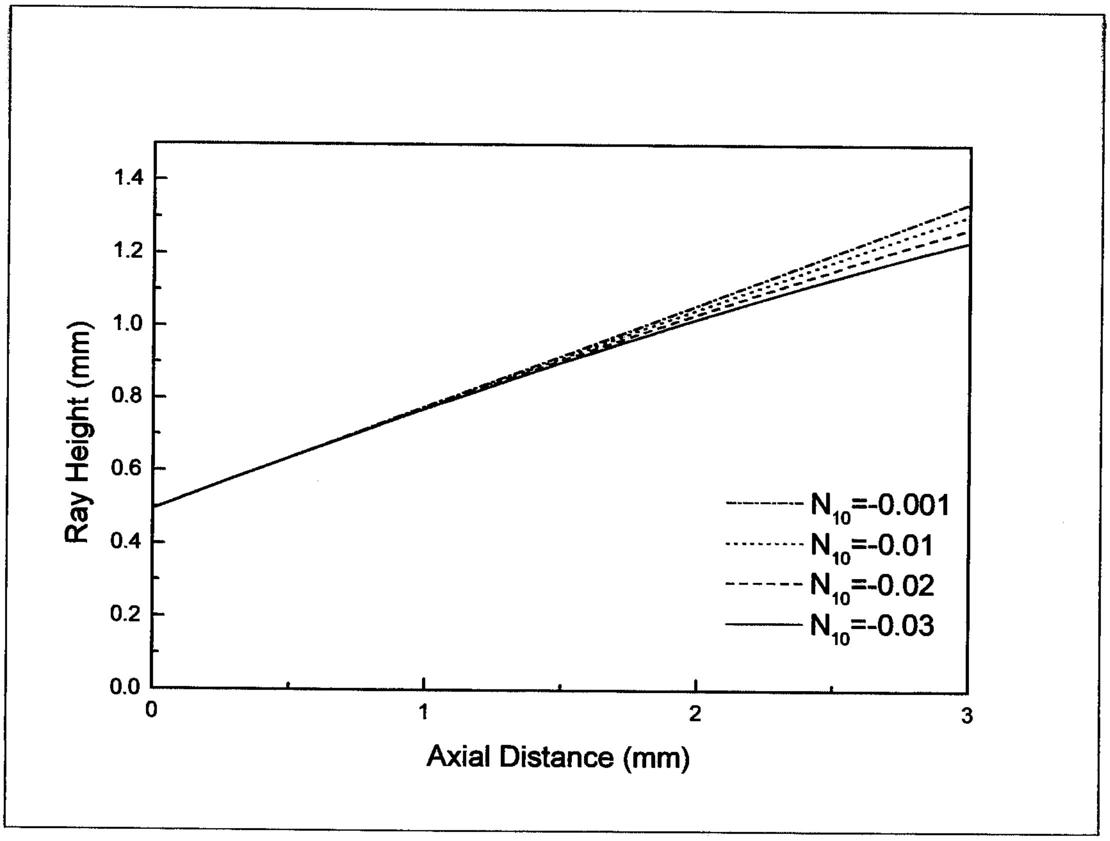


Fig.(3.15):Ray path through a radial GRIN lens at $\alpha = 10^0$.

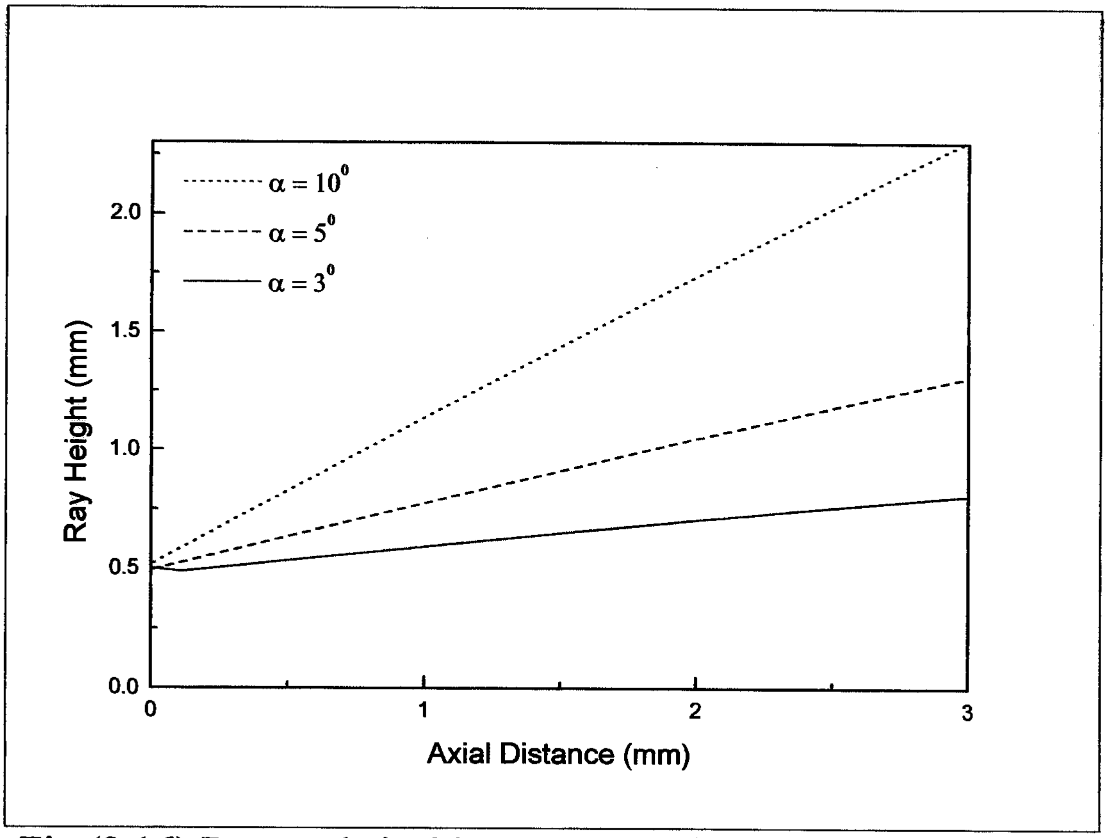


Fig.(3.16):Ray path inside a radial GRIN lens at N_{10} = -0.01 and at different angles.

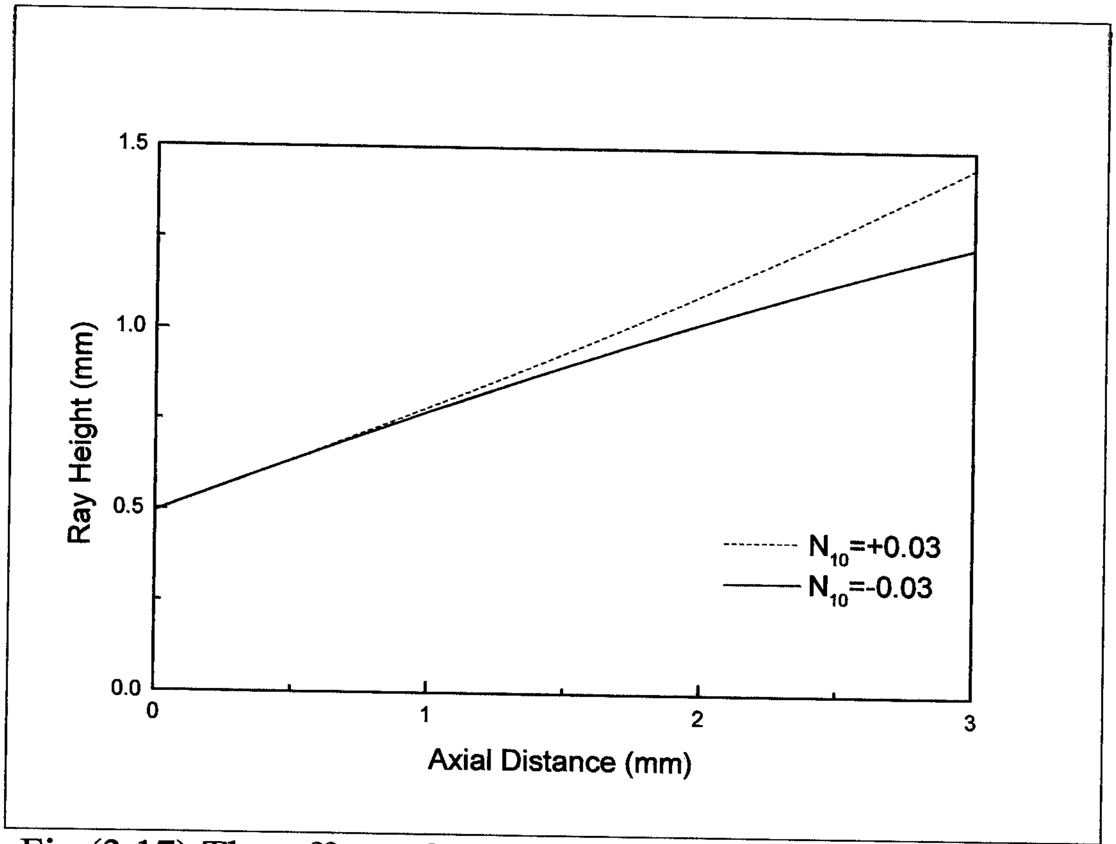


Fig.(3.17):The effect of gradient parameter's sign on ray path through a radial GRIN lens.

Chapter 4

LENS OPTIMIZATION BY USING GRIN MEDIUM.

In this chapter we are going to study the effect of GRIN lens on Transverese Spherical Aberration (TSA), as an application of using the GRIN materials for optimizing the optical performance of lenses. Therefore in the beginning we will represent a theoritical introduction for the aberration theory of optical system[10] , then we will use ZEMAX program to find TSA in an original lens (without gradient refractive index) and in a similar GRIN lens such as: a linear axial , a quadratic axial and a radial GRIN lens. Where we will use the same test lens

given in chapter 3. Finally we will compare the value of TSA for both lenses.

In the next section, we will introduce a brief of an optical aberration theory terminology.

4.1 *Optical Aberration Theory*:

Perferct image formation in an optical system occurs when all of the rays originating from a single object point cross at a single image point[2], or equivalently, when the geometrical wavefront in image space has a spherical shape centered on the image point . For such an optical system, there is no aberration but in fact, there is no ideal perfect optical system.

The well-known Gaussian formula[20] was developed using the approximations that the lens is very thin and the rays are close to being on-axis (paraxial rays). This approximation allows us to assume that rays are bent at one plane (the lens plane), the latter implies that we can use the approximation $\sin\theta \approx \theta$, which is known as first-order theory[12] ,(see Fig.4.1).

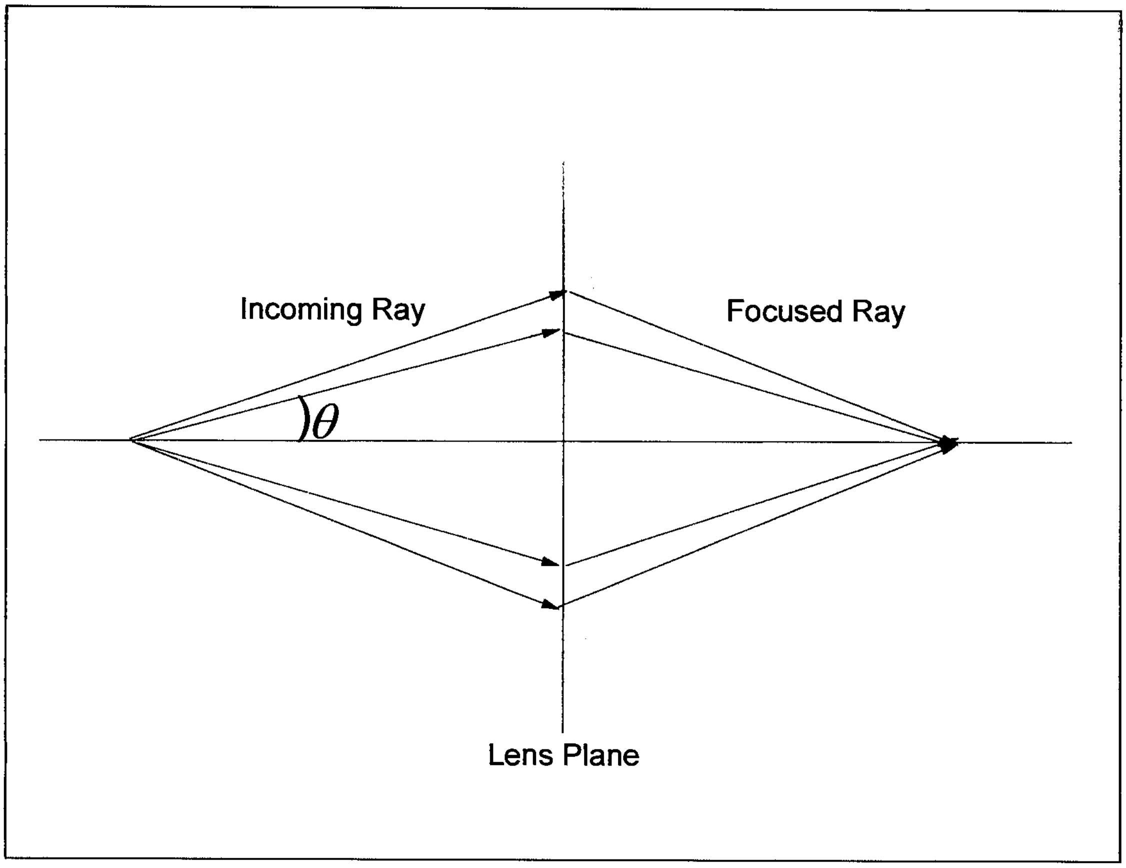


Fig.(4.1): Gaussian Approximation for paraxial rays.

These approximation lead to a mathematically tractable formula that can applied successfully in a wide range of applicatipons. The deviations from the Gaussian formula are called aberrations. While aberrations can be limited and, in certain cases, corrected for, it must be noted here that lens aberration is not simply the result of a poorly contrived lens. Lord Rayleigh[8] provided a mathematical proof that in order for an optical system to produce a perfect image of an object,(optimized system). This provision states that all of the optical path lengths (OPL's) for each possible ray connecting an object point to its corresponding image point (conjugate points), must be equal. This situation is never satisfied for a real optical system with spherical, or for oval and conic sections away from their unique pair of conjugate points; the only optical element that exhibits stigmatic image for all pairs of conjugate points is the plane mirror. This leads us to expect deviations from ideal (stigmatic)[1] imaging which we refer to as aberrations. Optical aberrations are highly dependent upon surface curvature,

conjugate ratios, aperture stop location and diameter, and field angles.

Optical aberration theory has been used for gaining a deeper insight the performances and limitations of optical systems through the lens design process. Even if presently only third-order where, $sin\theta \approx \theta - \frac{\theta^3}{3!}$,(Seidel)[16] coefficients are widely used in practical applications, it is known that for the design of high-quality optical systems higher order aberration coefficients can be a powerful tool. Intuitively, optical systems are often desinged in such a way that the ray paths inside the system are as "relaxed" as possible, i.e. the incident angles, ray hieghts and slopes tend to be everywhere as small permitted by the aperture and field requirements. Any surface for which these ray parameters are too large can be a major source of aberrations. The types of aberrations can be classification as: *Monchromatic aberration*[2] and *Chromatic aberration*[2] , arise due to geometrical deviations paraxial (Gaussian) theory. The five primary or Seidel aberrations are: spherical aberration, coma,

astigmatism, field curvature, and distortion. While chromatic aberrations results from the dependence of index of aberration on wavelength ($n \to n(\lambda)$). This causes the properties of an optical system, such as magnification, focal length, and location of all principal points, to be different for each color of light passing through the system. Diffractive aberrations[10] are caused by deviations from geometrical optics due to the wave nature of light. Diffraction effects place the ultimate limit on a system's best possible imaging performance.

In next section we are going to represent a brief about monochromatic aberration.

4.2 *Monochromatic Aberration:*

4.2.1 *Spherical Aberration:*

Spherical aberration[12] is the name given to the effect where the focal length of lens will vary depending on how far you are from the center of the lens as in figure (4.2). The distance along the optical axis between the intercept of the rays that are nearly on the optical axis (paraxial rays) and the rays

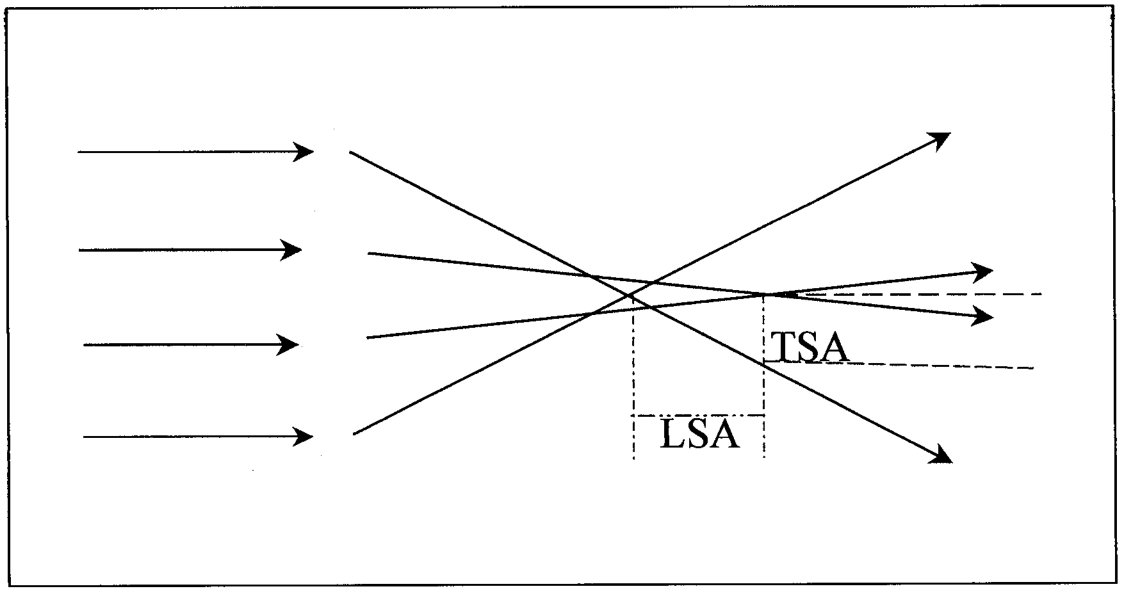


Fig.(4.2): Longitudinal Spherical Aberration(LSA) and Transverse Spherical Aberration(TSA).

that go through the edge of the lens (marginal rays) is called longitiudnal spherical aberration (LSA). The distance at which these rays intercept the paraxial focal plane, mesured from the optical axis, is called transverese spherical aberration (TSA). Spherical aberration depends on lens shape, orientation, and conjugate ratio. The simplest way to eliminate or reduce spherical aberration is to make the lens surface with a varying radius of curvature (i.e., an aspheric surface). This lens is desinged to exactly compensate for the fact of first-order theory at large angles. In practice, however, most lenses with high surface quality are manufactured by grinding and polishing techniques that naturally produce spherical or cylindrical surface. The manufacture of aspheric surfaces is more complex, and it is difficult to produce a lens of sufficient surface accuracy to eliminate spherical aberration completely. Fortunately, these aberrations can be virtually eliminated, for a chosen set of conditions, by combining the effects of two or more spherical (or cylindrical) surfaces. In general, simple positive lenses have undercorrected[28] spherical aberration and negative lenses

usually have overcorrected[28] spherical aberration. By combining a positive lens made of low-index glass with negative lens, it is possible to produce an optical system in which the spherical aberrations is minimized.

4.2.2 Coma:

In spherical lenses, different parts of the lens surface exhibit different degrees of magnification. This gives rise to an aberration known as coma[2]. As shown in figure (4.3), each concentric zone of the lens forms a ring-shaped image called a comatic circle. This cause blurring in the image plane for off-axis[12] object points. An off-axis object point will not have sharp image point, but it appears as a characteristic comet-like flare. As with spherical aberration, correction can be achieved by using multiple surfaces. Alternatively, a sharper image may be produced by judiciously placing an aperture, or stop, in an optical system to eliminate the more mariginal rays.

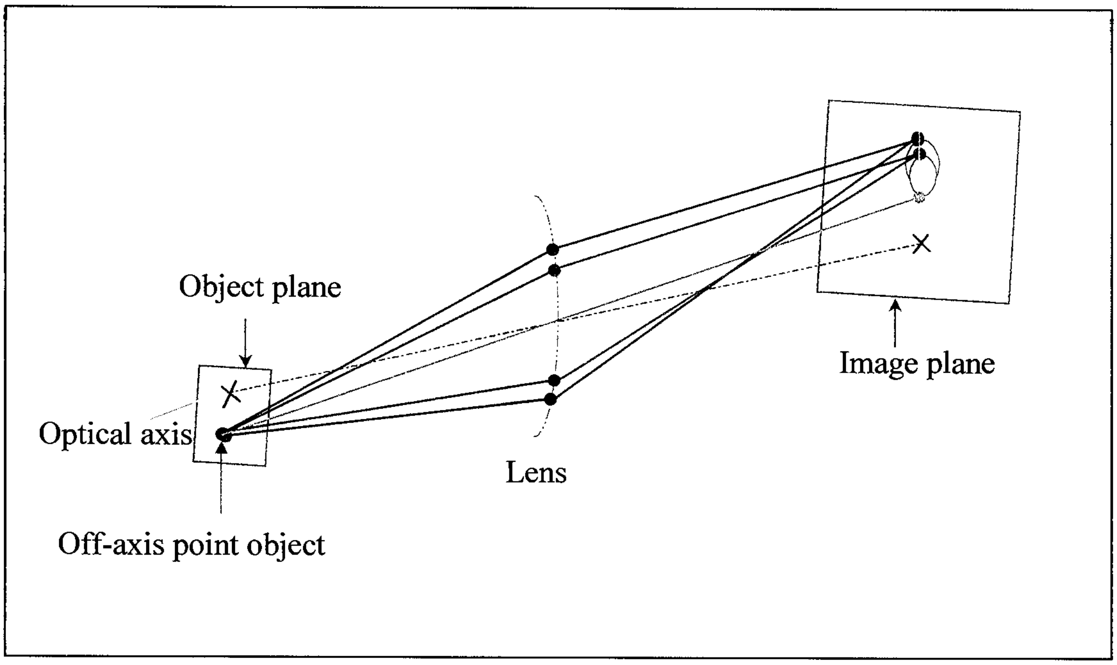


Fig.(4.3): imaging of an off-axis point source by a lens with positive transverse coma. Each cone of rays passing through a circle of the lens surface is imaged not to a point, but as a comatic circle.

4.2.3 *Astigmatism:*

When an off-axis object is focused by a spherical lens, the natural asymmetry leads to astigmatism[28]. As shown in figure (4.4), the plane containing both optical axis and object point is called the tangential plane[12]. Rays that lie in this plane are called tangential rays[12]. Rays not in this plane are referred to as skew rays. The chief, or principal, rays goes from the object point through the center of the aperture of the lens system. The plane perpendicular to the tangential plane that contains the principal ray is called the sagittal or radial plane. Figure (4.4) illustrates that tangential rays from the object come to a focus closer to the lens than do rays in sagittal plane. When the image is evaluated at the tangential conjugate, we see a line in the sagittal direction. A line in the tangential direction is formed at the sagittal conjugate. Between these conjugates, the image is either an elliptical or circular blur. Astigmatism is defined as the separation of these conjugates. The amount of astigmatism in a lens depends on lens shape only when there is an aperture in the system that is not in contact with the lens itself. (In all optical

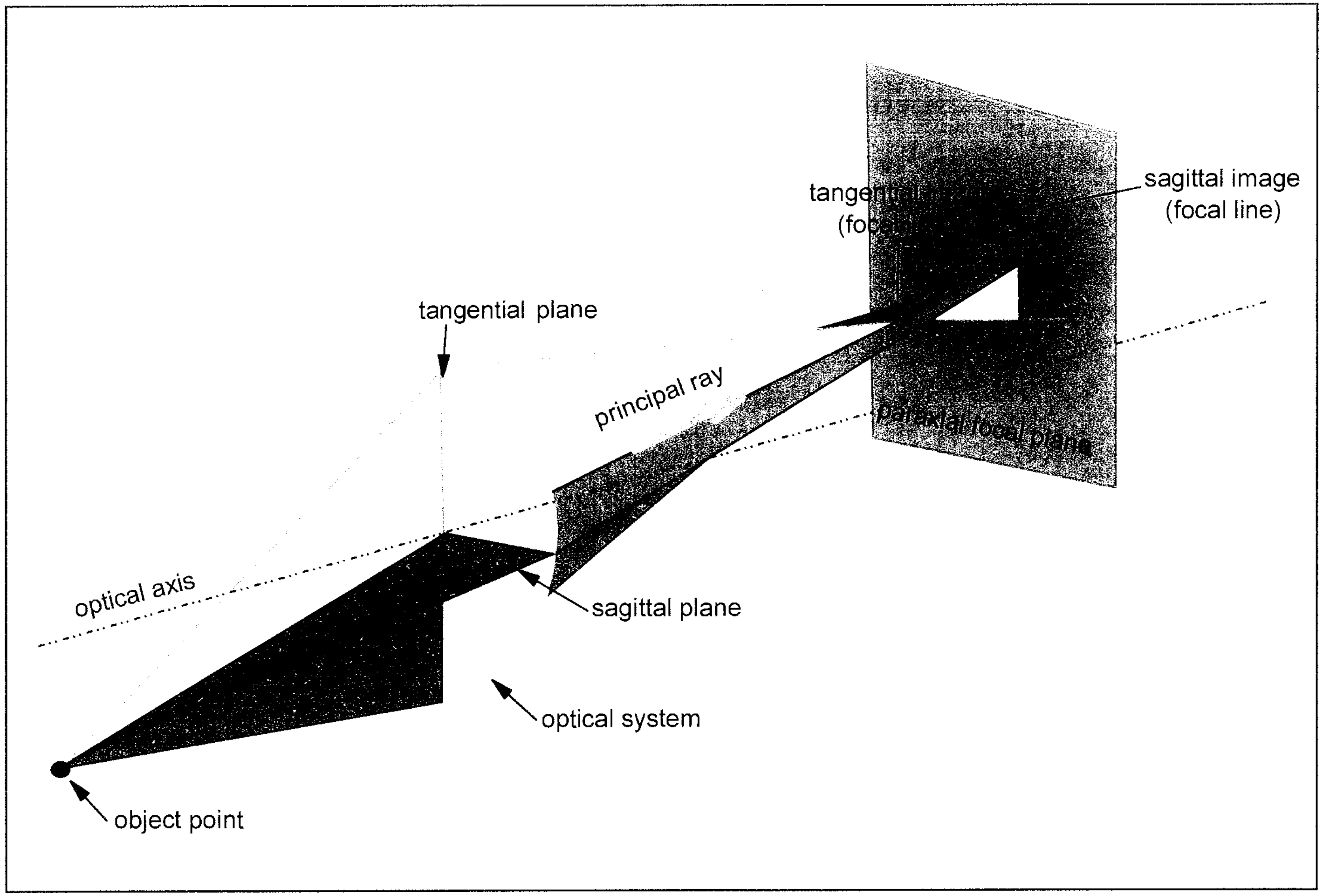


Fig.(4.4): Astigmatism represented by sectional views.

system there is an aperture or stop, although in many cases it is simply the clear aperture of the lens element itself). Astigmatism strongly depends on the conjugate ratio.

4.2.4 *Field Curvature:*

Even in the absence of astigmatism, there is a tendency of optical system to image better on curved surfaces than on flat planes. This effect is called field curvature(2), (see Fig. 4.5). In the presence of astigmatism, this problem is compounded because there are two separate astigmatic focal surfaces that correspond the tangential and sagittal conjugates. Field curvature varies with the square of field angle. Therefore, by reducing the field angle by one-half, it is possible to reduce the blur from the field curvature to a value of 0.25 of its original size. Positive lens elements usually have inward curving fields, and negative lenses have outward curving field curving fields. Field curvature can thus corrected to some extent by combining positive and negative lens element.

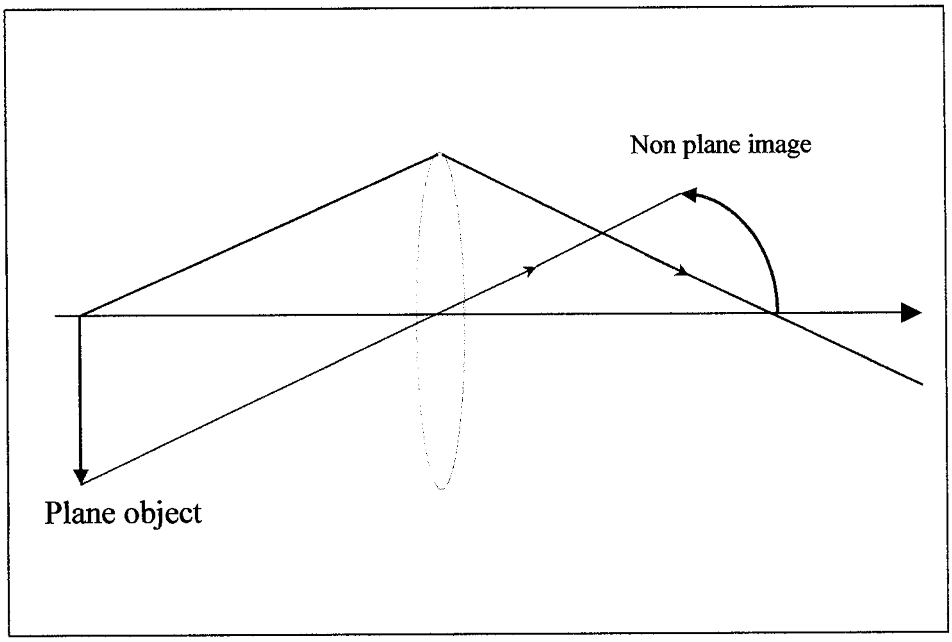


Fig.(4.5): Curvature of field.

4.2.5 *Distortion:*

The image field not only may have curvature, but may also be distorted. The image of an off-axis point may be formed at a location on this surface other than the predicted by the simple paraxial equations. This distortion differes from coma (where rays from an off-axis fail to meet perfectly in the image plane). Distortion means that even if a perfect off-axis point image is formed, its location on the image plane is not correct. Furthermore, the amount of distortion usually increasing image hight. The effect of this can be seen as two different kinds of distortion : pincushion and barrel, see figure (4.6). Distortion is a separation of the actual image plane and can be expressed either as an absolute value or as a percentage of the paraxial image height.

In next section we are going to represent a brief about chromatic aberration.

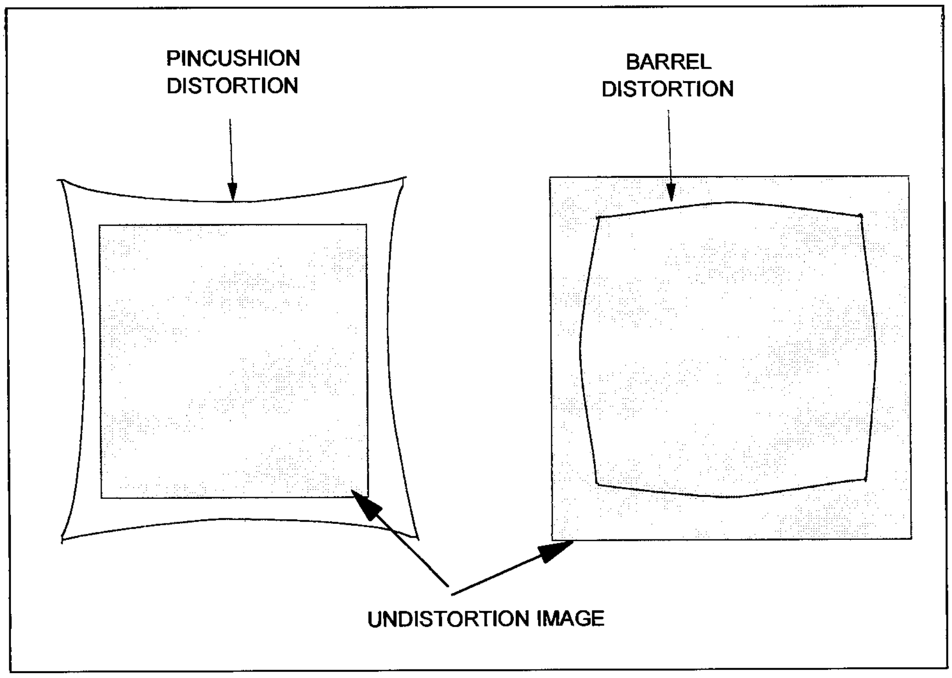


Fig.(4.6): Distortion

4.3 *Chromatic Aberration*:

This type arise when lenses are used to transform light containing multiple wavelengths. Unfortunately for len desinger, glasses do not slow down of chromatic light by the same amount. Instead, the blue light, with a short wavelength, is slowed down more than red light, see figure (4.7), so the glass has a higher index of refraction for blue light than for red. There are two forms of chromatic aberration, axial (or longitidinal)[6] chromatic aberration and lateral (or transverse)[6] chromatic aberration. As in the case of spherical aberration, positive and negative elements have opposite signs of chromatic aberration. By combining elements of nearly opposite aberration to form a doublet, chromatic aberration can be partially corrected.

In next section we will compare the amount of the transverese spherical aberration in an original lens with a similar lens made of axial quadratic and radial GRIN lenses.

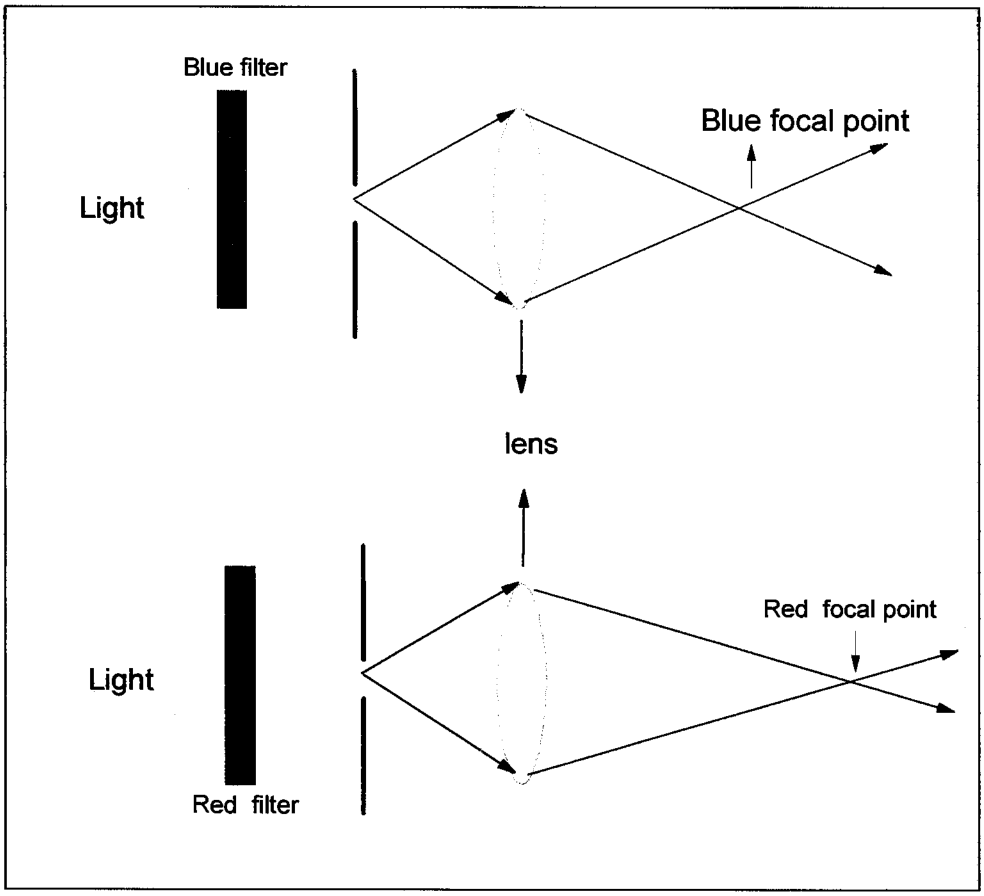
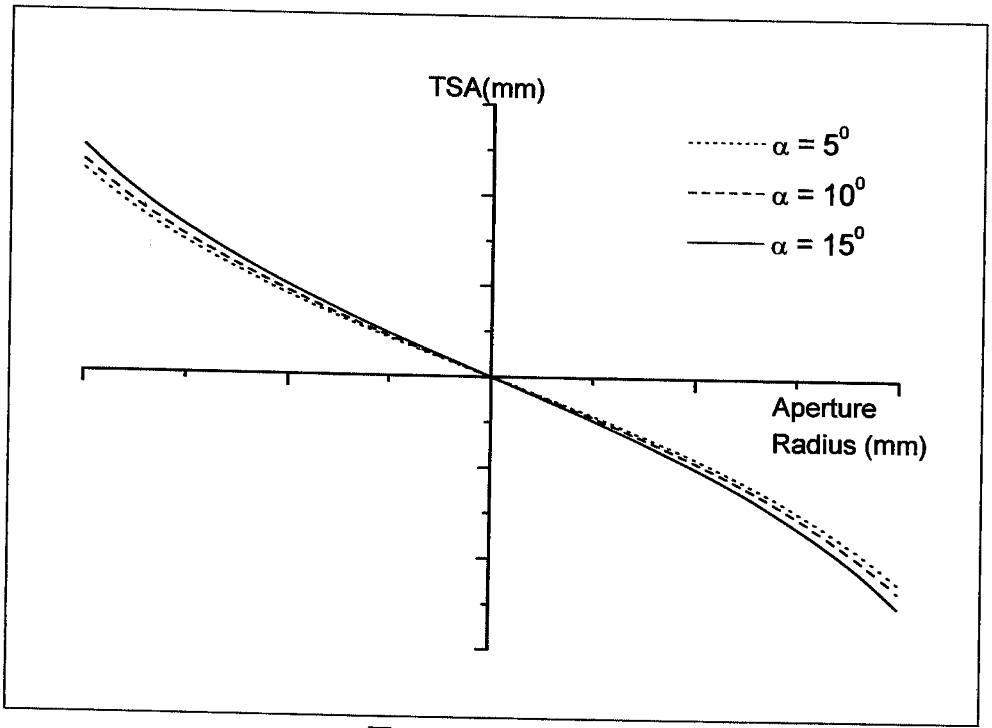


Fig.(4.7): Chromatic aberration: different focal lengths for blue light(top) and red light(bottom).

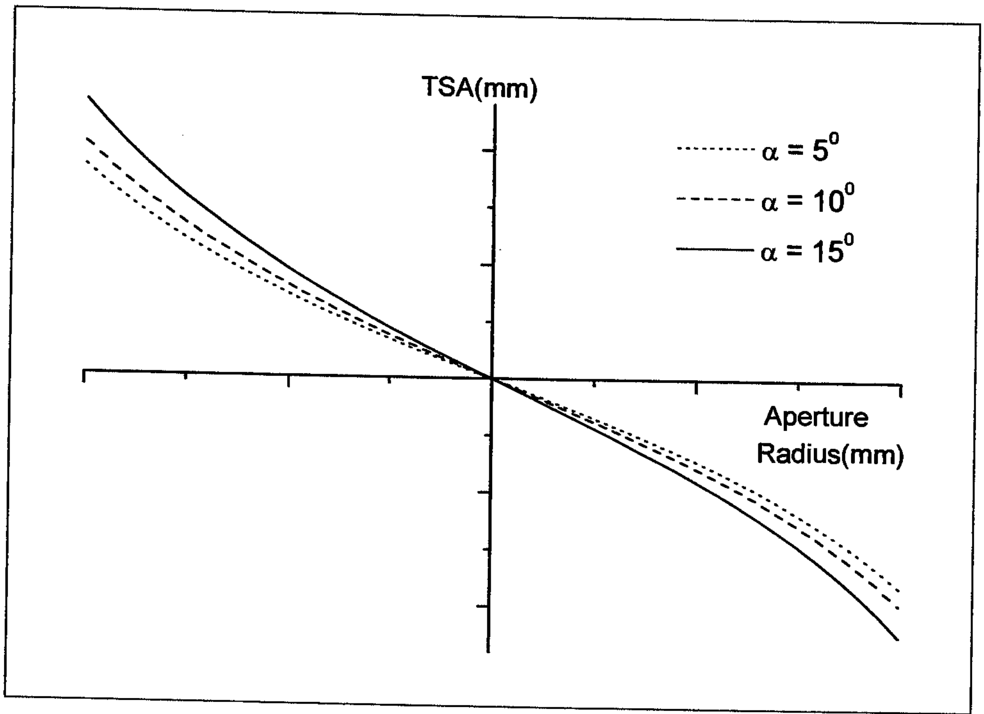
4.3 Comparision of transverse ray aberration in a homogeneous lens and GRIN lens:

A classical application of the use of GRIN lenses is to correct a spherical aberration. The spherical aberration of an optical system is usually represented graphicaly[28], and it can be defind as the variation of focus with aperture. TSA is the departure of a particular ray from the paraxial image point, measured in the focal plane of the optical system, in another word it is the name given to the aberration when it is measured in the vertical direction. For a given aperture , focal length, and by using the same parameter of plano-convex lens ,(see chapter 3), TSA is plotted as a function of aperture radius for a several incident angle by helping of ZEMAX program.

Practically, we will study TSA in two ways. The first way, how the variation of angles of incidence can be effect on TSA , the second way, how the GRIN lens can be effect on the amount of TSA. For the first way, the amount of TSA as a function of aperture radius is graphed for different angles such as (5^0, 10^0 and 15^0) for tangential and sagittal fans. In Fig. (4.8), an original



Tangential fan

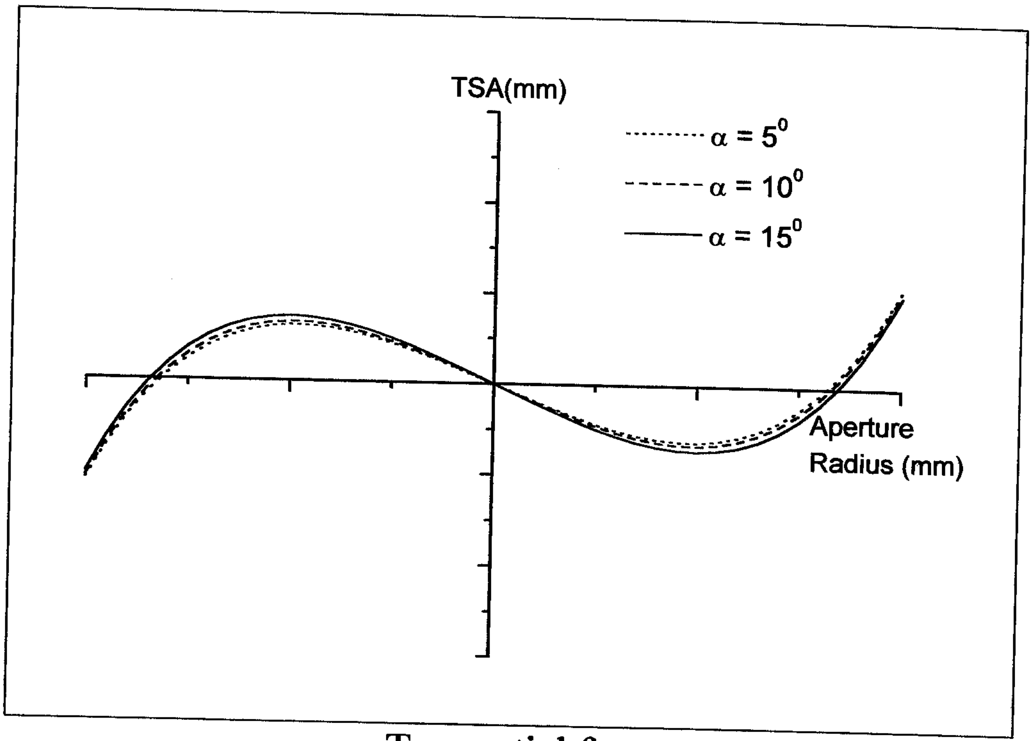


Sagittal fan

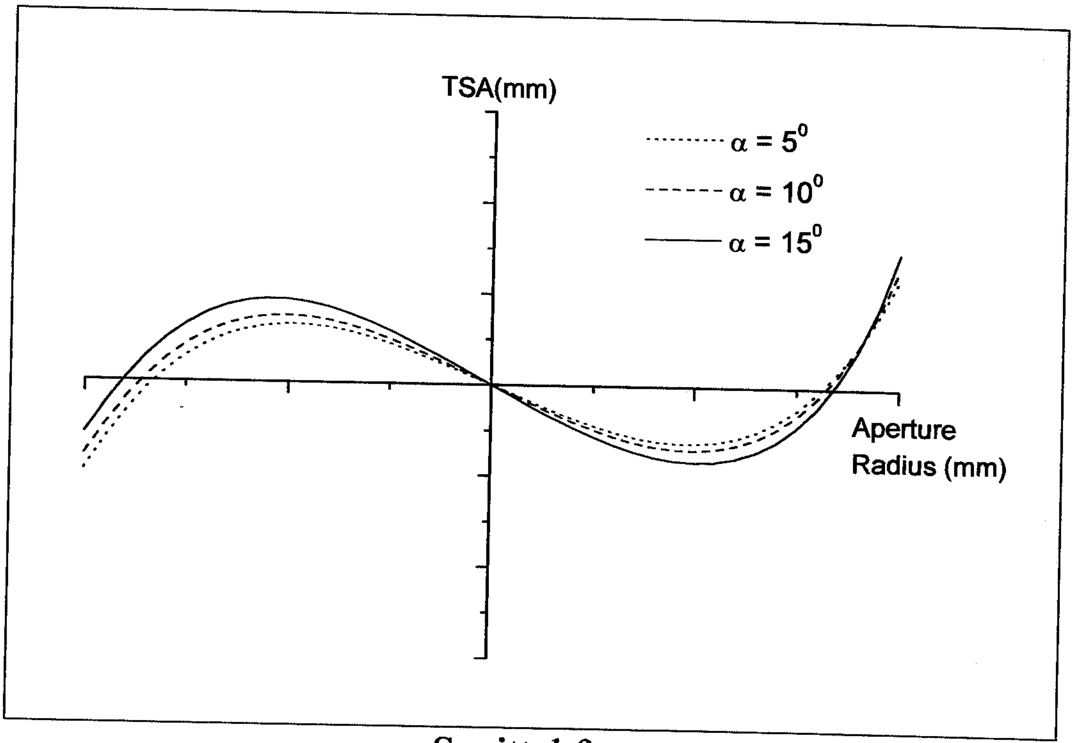
Fig.(4.8): Transverse spherical aberration in an original lens at different angles.

lens (homogeneous lens) is used, in Fig. (4.9) we use a similar lens of a linear axial GRIN lens and in Fig.(4.10) we use the same similar lens with a quadratic axial GRIN lens. Finally we use a similar lens with a radial GRIN lens and graphed the amount of TSA as a function of aperture radius in different angles such as (5^0, 10^0 and 15^0), (see Fig. 4.11).

In the second way, to know how the types of the lens can be effect on TSA, the amount of TSA will be compare between its value in an original lens (with out gradient index) and in a GRIN lens (axial and radial). These amounts is graphed as a function of aperture radius, where Fig.(4.12) represents the comparision of TSA values as a function of aperture radius between the original lens and its equivalent lens with a linear axial GRIN material at $\alpha = 10^0$, where we can see that if we use an axial GRIN lens, the values of TSA are reduced, and in Fig.(4.13), the comparision of TSA values as a function of a perture radius between the original lens and its equivalent lens made of a quadratic axial GRIN materials is represented at $\alpha = 10^0$.



Tangential fan



Sagittal fan

Fig.(4.9): Transverse spherical aberration in a linear axial GRIN at different angles.

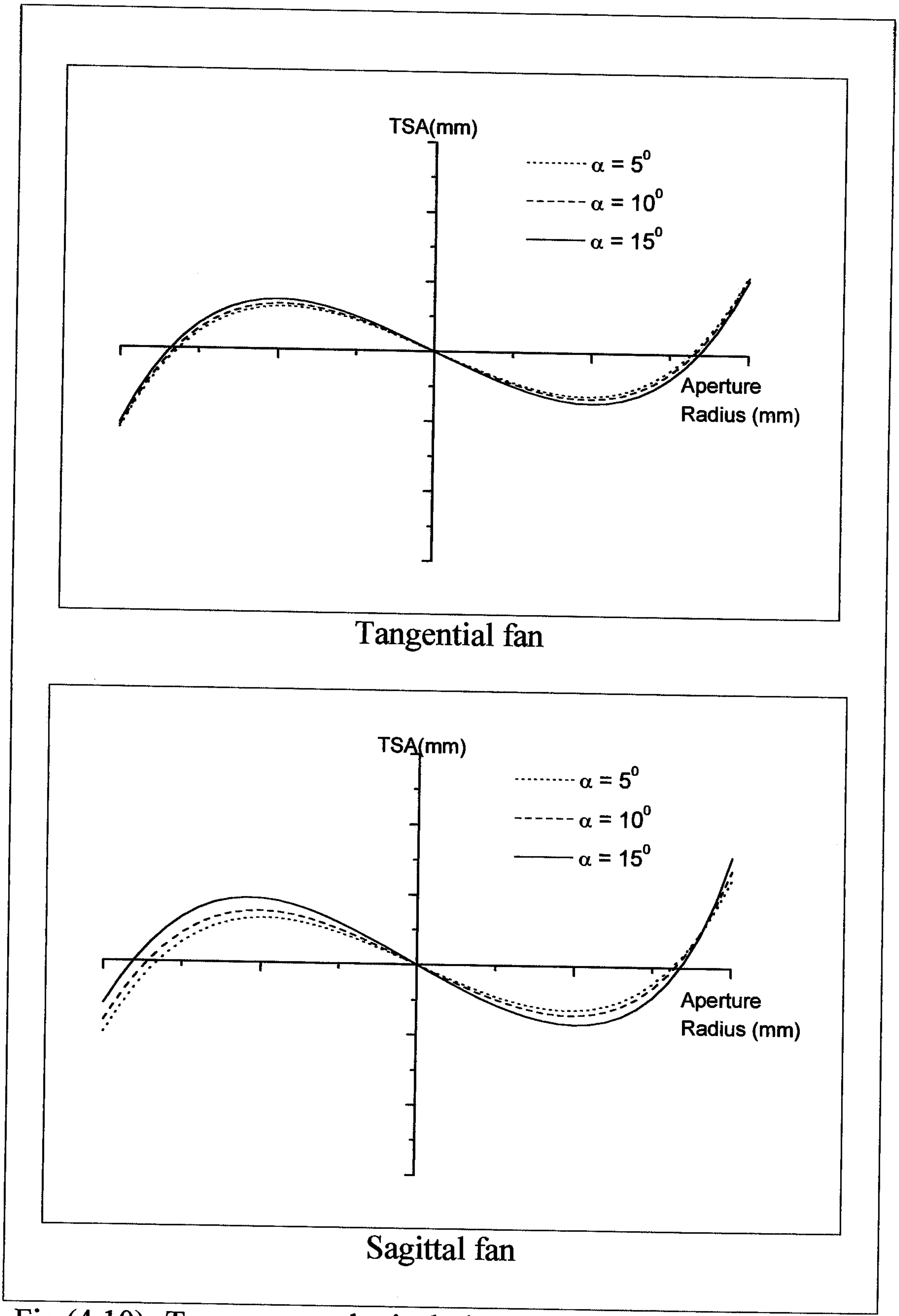


Fig.(4.10): Transverse spherical aberration in a quadratic axial GRIN lens at different angles.

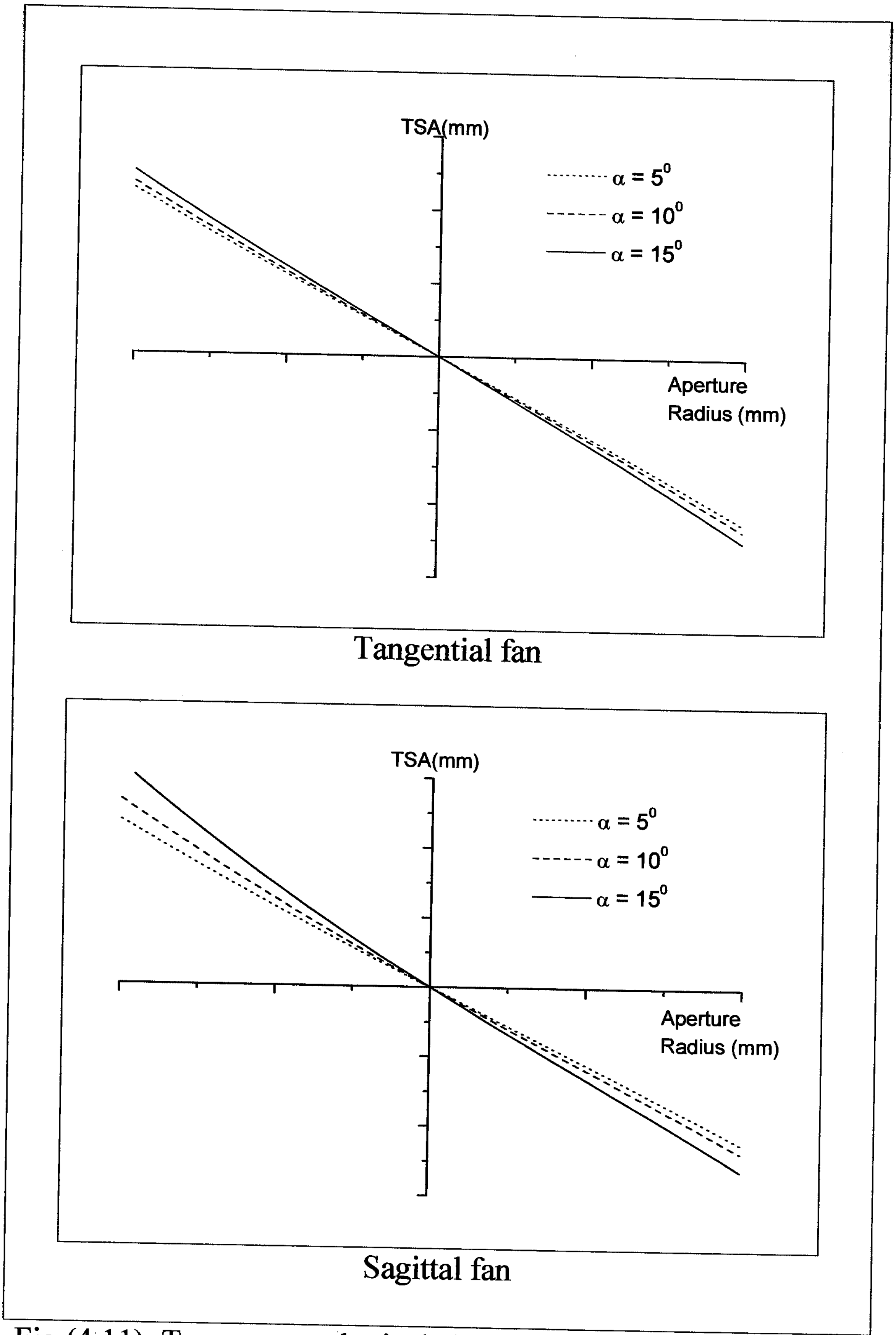
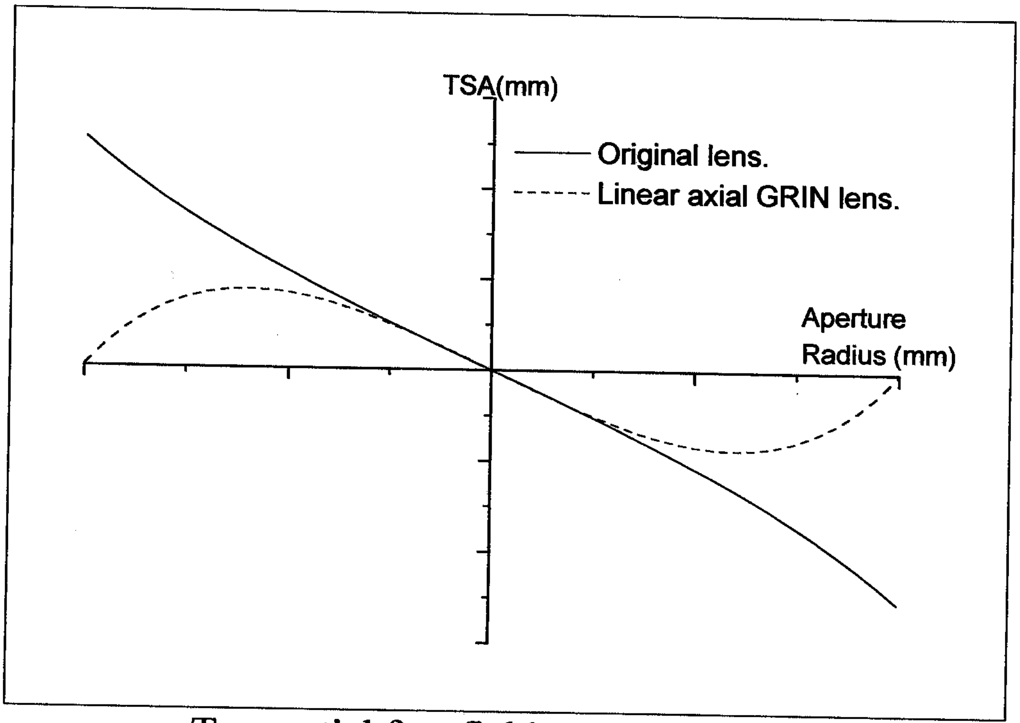
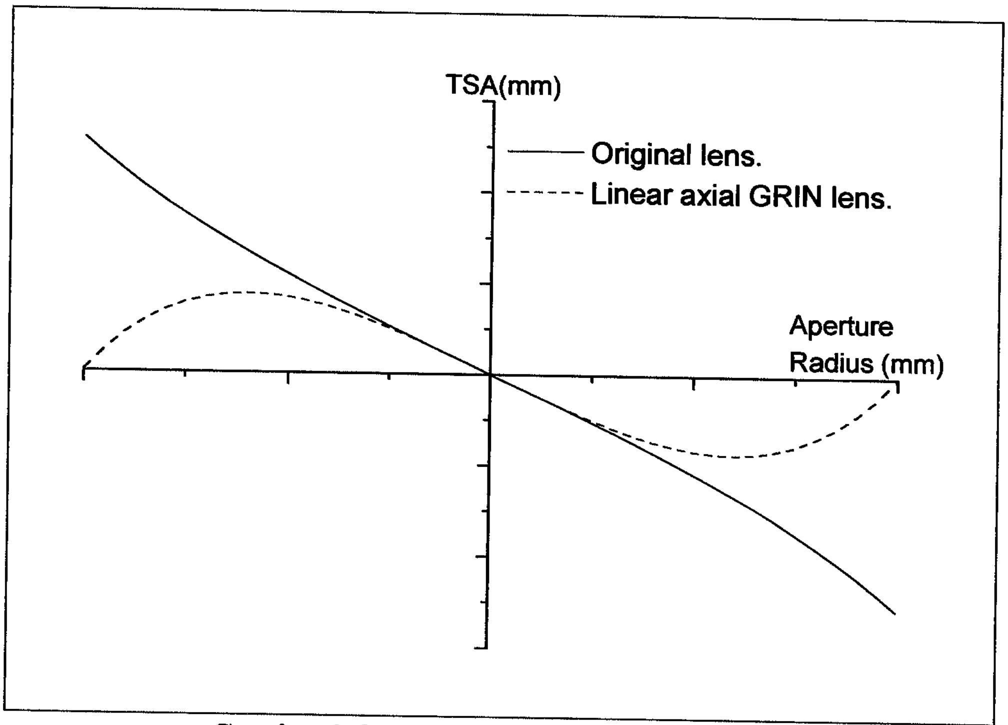


Fig.(4.11): Transverse spherical aberration in a radial GRIN lens at different angles.

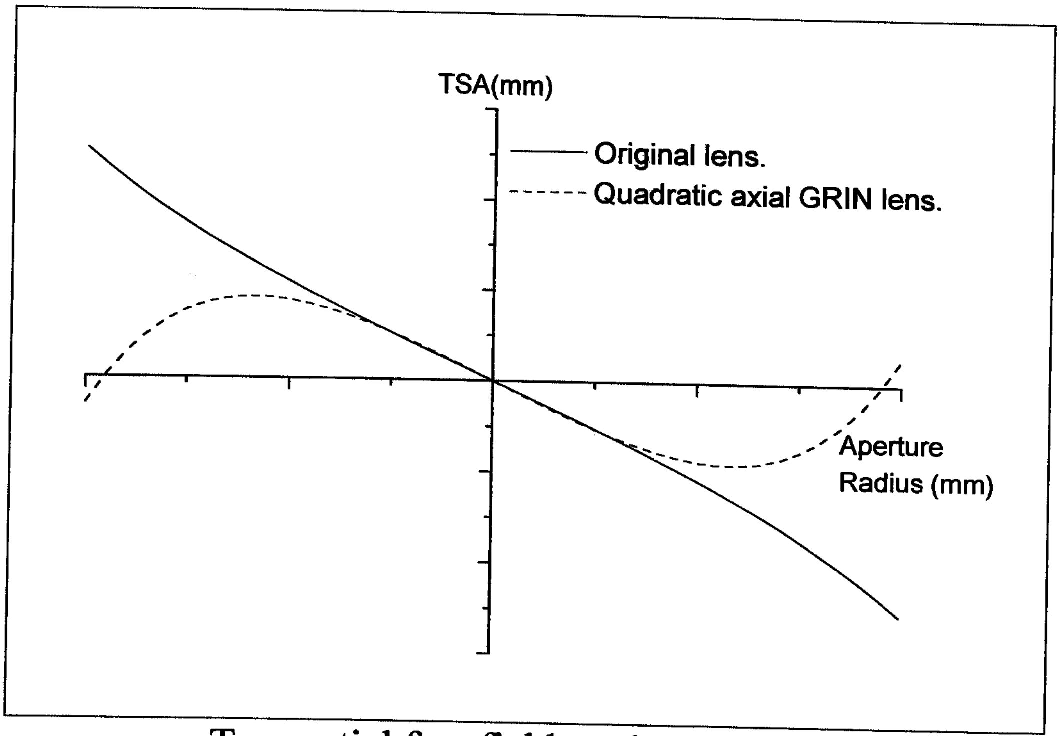


Tangential fan, field angle = 10 Deg.

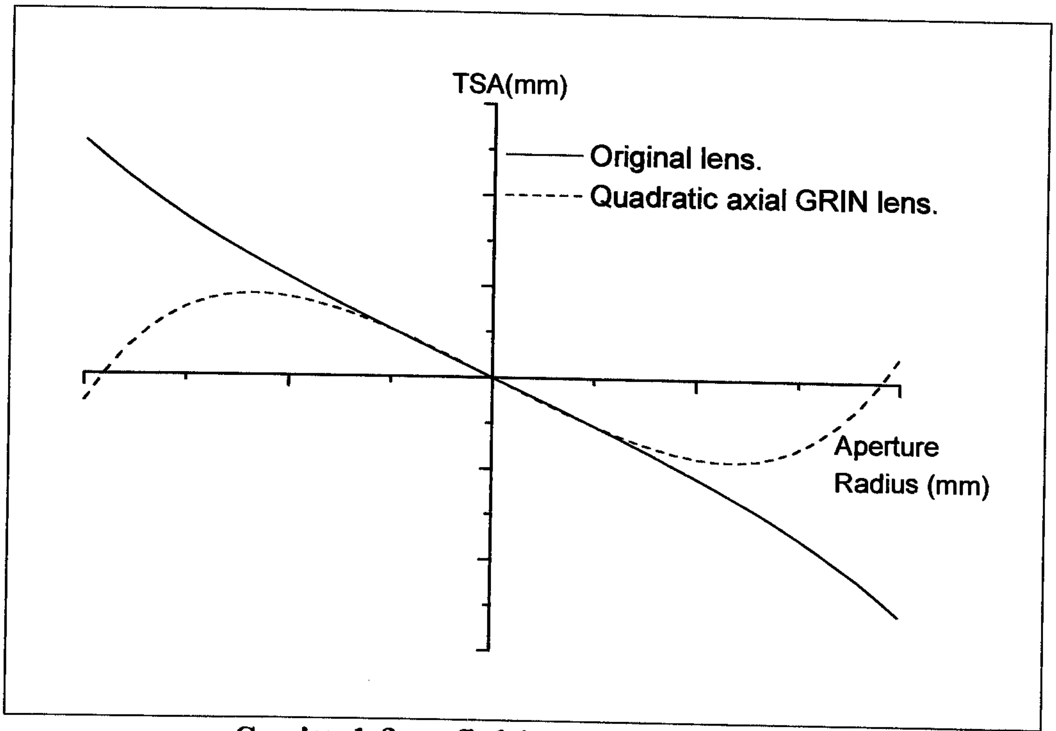


Sagittal fan, field angle = 10 Deg.

Fig.(4.12): Transverse spherical aberration in an original lens and a linear axial GRIN lens.



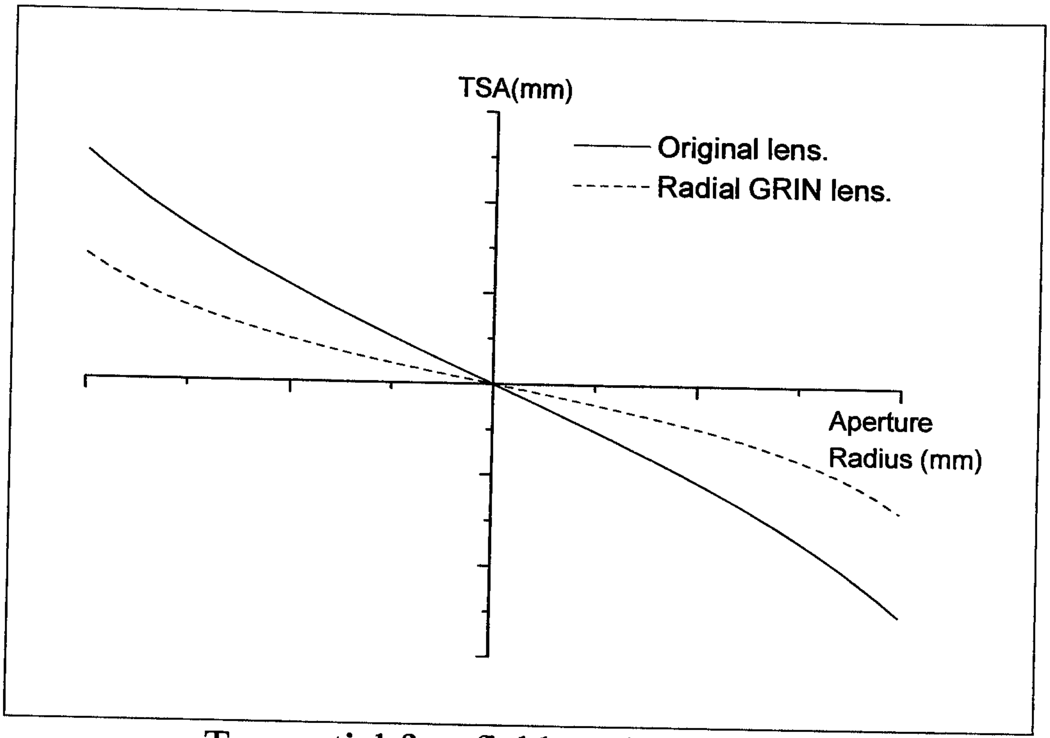
Tangential fan, field angle = 10 Deg.



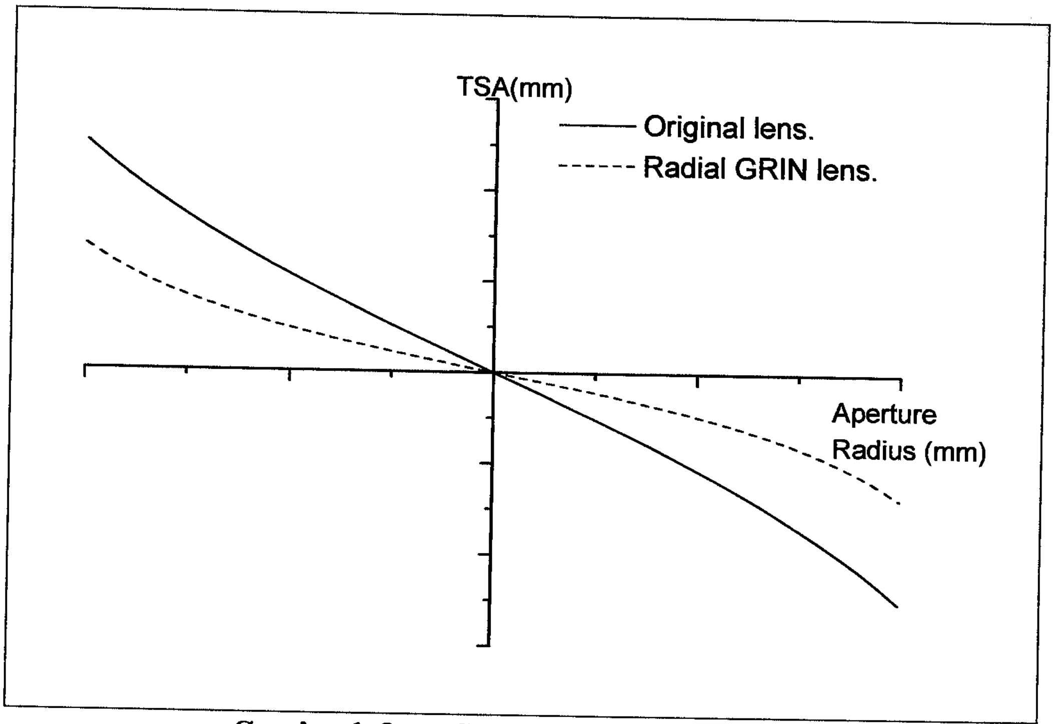
Sagittal fan, field angle = 10 Deg.

Fig.(4.13):Transverse spherical aberration in an original lens and a quadratic axial GRIN lens.

Finally Fig.(4.14) represents the comparision of TSA values as a function of aperture radius between the original lens and its equivalent lens made of a radial GRIN materials.



Tangential fan, field angle = 10 Deg.



Sagittal fan, field angle = 10 Deg.

Fig.(4.14): Transverse spherical aberration in an original lens and a radial GRIN lens.

Chapter 5

SUMMARY AND CONCLUSION

It is readily apparent that Sharma's method of ray tracing as outlined in chapter 2 of this study, does provide additional information about the path of the ray by using GRIN lens which enables one to help improve performance or reduce complexity in optical system. This method essentially consists of transforming the ray equation into convenient form and solving the resulting equation using a standard numerical technique , Runge-Kutta method. Ray tracing predicts the direction cosine of the ray and its position at any point inside lens. By using these information and by using a GRIN lens (axial and radial) ,

the form of ray path can be described. For example, this path takes a curve line in different shapes , i.e., in a linear axial GRIN, the path takes a curve line close to a straight line while in a quadratic axial GRIN this path takes a line with tilt more than the first one , but in a radial GRIN (with order up to two) the path takes a curve line clearly. In view of this study, it is clear that the ray tracing , which based on numerical technique , is a very powerfull method for lens design.

The usefull of GRIN lens is to optimize the optical aberration of such system. In this study, the comparision of TSA in axial GRIN and radial GRIN represent that , radial GRIN enjoy a significant advantage over axial GRIN for small values of aperture radius.

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Appendix A

THE LAW OF REFRACTION

Consider a ray of light that is refracted at an interface between two homogeneous materials characterized by the indices of refraction of materials n_0,n_1 respectively, $n_1 > n_0$, (see Fig. A.1). In this figure the interface is curved surface. The incident ray $\boldsymbol{T_0}$, intersects the boundary at a point '*a*' , around this point, two arcs whose radii are proportional to n_0 and n_1 are drown. $\boldsymbol{T_0}$ is extented until it intersects the n_0 arc at '*b*'. Then the line $\boldsymbol{N}$, is drawn normal to the boundary, therefore line $\boldsymbol{N}$ is a radius of the surface connecting the point of incident '*a*' and the center of curvature '*c*'.

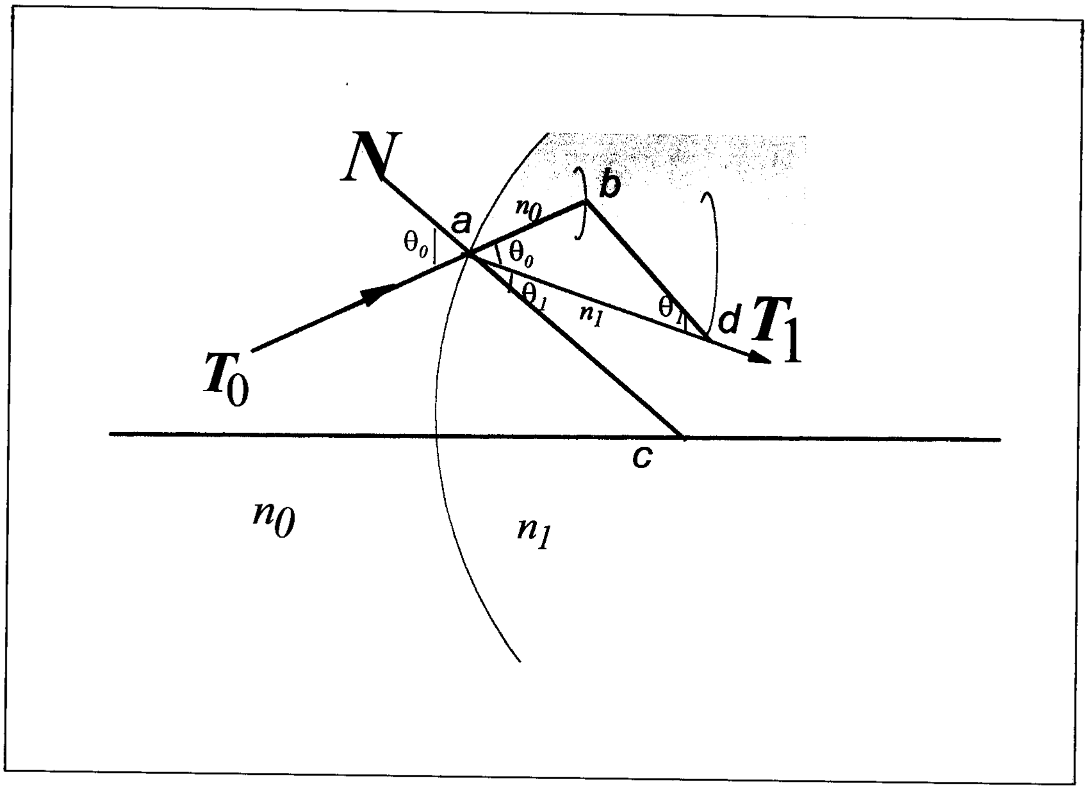


Fig.(A.1): Ray tracing through two media.

Then draw line which parallel to $\boldsymbol{N}$, which intersect the n_1 arc at 'd'. The refracted ray can be found by connecting 'a' with 'd'. The incident rays ,$\boldsymbol{T_0}$, and the refracted ray ,$\boldsymbol{T_1}$, can be written as:

$$\boldsymbol{T}_0 = n_0 \boldsymbol{A}_0 \tag{A.1a}$$

$$\boldsymbol{T}_1 = n_1 \boldsymbol{A}_1 \tag{A.1b}$$

where $\boldsymbol{A_0}$, $\boldsymbol{A_1}$ are unit vectors along the incident and refracted ray.

From triangle 'abd' in Fig.(A.1) we have:

$$\frac{|ab|}{sin\theta_1} = \frac{|ad|}{sin(\pi - \theta_0)}$$

since:

$$|ab| = |\boldsymbol{T}_0| = |n_0 \boldsymbol{A}_0| = n_0 \quad , \quad |ad| = |\boldsymbol{T}_1| = |n_1 \boldsymbol{A}_1| = n_1 \tag{A.1c}$$

then,

$$\frac{n_0}{sin\theta_1} = \frac{n_1}{sin\theta_0}$$

therefore:

$$n_0 sin\theta_0 = n_1 sin\theta_1 \tag{A.2}$$

Eq.(A.2) represent Snell's law.

The vectorial form of Eq.(A.2), can be derived as follow:

Referring to Fig.(A.1), we have from triangle '*abd*' :

$\overline{ad} = \overline{ab} + \overline{bd}$

therefore:

$$\boldsymbol{T_1} = \boldsymbol{T_0} + \Omega \boldsymbol{N} \qquad \text{(A.3)}$$

To determine the scalar quantity Ω, multiply both sides of Eq.(A.3), by $\boldsymbol{T_0}$ (scalar product):

$$\boldsymbol{T_1} \cdot \boldsymbol{T_0} = \boldsymbol{T_0} \cdot \boldsymbol{T_0} + \Omega \boldsymbol{N} \cdot \boldsymbol{T_0} \qquad \text{(A.4)}$$

by helping of Eq.(A.1c), Eq.(A.4) becomes:

$$\boldsymbol{T_1} \cdot \boldsymbol{T_0} = n_0^2 + \Omega(\boldsymbol{N} \cdot \boldsymbol{T_0}) \qquad \text{(A.5)}$$

from Eq.(A.3):

$$\boldsymbol{T_1} - \boldsymbol{T_0} = \Omega \boldsymbol{N}$$

by squaring both sides:

$$(\boldsymbol{T_1} - \boldsymbol{T_0})^2 = \Omega^2 (\boldsymbol{N})^2$$

$$T_1^2 - T_0^2 - 2(\boldsymbol{T_1} \cdot \boldsymbol{T_0}) = \Omega^2 \qquad \text{(A.6)}$$

by substituting from Eqs(A.1c) and (A.5) into Eq.(A.6), we have:

$$\Omega^2 + 2(\boldsymbol{N} \cdot \boldsymbol{T_0})\Omega - (n_1^2 - n_0^2) = 0 \qquad \text{(A.7)}$$

Eq.(A.7) represent a second order equation, its solution take the following form:

$$\Omega = -(\boldsymbol{N}\cdot\boldsymbol{T}_0) \pm \sqrt{(\boldsymbol{N}\cdot\boldsymbol{T}_0)^2 + (n_1^2 - n_0^2)}$$

let $v = (\boldsymbol{N}\cdot\boldsymbol{T}_0)$, (v scalar functions).

Therefore:

$$\Omega = -v \pm \sqrt{n_1^2 - n_0^2 + v^2} \qquad \text{(A.8)}$$

Referring to Eq.(A.1a) and Eq.(A.1b), we can write in general:

$$\boldsymbol{T} = n\boldsymbol{A}, \qquad \text{(A.9)}$$

where: $\boldsymbol{A}$ is a unit vector along the ray.

By taking the scalar value of Eq.(A.9) , we get:

$$\sqrt{T_x^2 + T_y^2 + T_z^2} = \sqrt{n^2}$$

Therefore:

$$T_x^2 + T_y^2 + T_z^2 = n^2 \qquad \text{(A.10)}$$

To satisfy Eq.(A.10), we must take the positive root in Eq.(A.8), so we can get:

$$\Omega = -v + \sqrt{n_1^2 - n_0^2 + v^2}\,. \qquad \text{(A.11)}$$

Therefore the vectorial form of Snell's law, Eq.(A.3), can be written as :

$$\boldsymbol{T}_1 = \boldsymbol{T}_0 + [-\nu + \sqrt{n_1^2 - n_0^2 + \nu^2}]\boldsymbol{N}$$

Appendix B

RUNGE-KUTTA METHOD

Runge-Kutta methods can be used to generate the solution of second order differential equation (ray equation). In this appendix we will give a summary of how to derive some Runge-Kutta shemes.

The basis of all Runge-Kutta methods is to express the difference between the values of y at x_{n+1} and x_n as:

$$y_{n+1} - y_n = \sum_{i=1}^{m} w_i k_i \tag{B.1}$$

where the w_i's are constant and:

$$k_i = h_n f(x_n + \alpha_i h_n, y_n + \sum_{j=1}^{i-1} \beta_{ij} k_j) \qquad \text{(B.2)}$$

with $h_n = x_{n+1} - x_n$ and $\alpha_1 = 0$ (B.3)

the expansion of L.H.S of Eq.(B.1) is:

$$y_{n+1} = y_n + \sum_{t=1}^{\infty} \frac{h_n^t}{t!} y_n^{(t)} \qquad \text{(B.4)}$$

where:

$$y_n^{(t)} = \frac{d^{t-1}}{dx^{t-1}} f(x_n, y_n) = (\frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y})^{t-1} f(x_n, y_n)$$

$$= (\frac{\partial}{\partial x} + f\frac{\partial}{\partial y})^{t-1} f(x_n, y_n) \qquad \text{(B.5)}$$

where $f = f(x, y)$.

By using Eq.(B.5) ,Eq (B.4) becomes:

$$y_{n+1} = y_n + \sum_{t=0}^{\infty} \frac{h_n^{t+1}}{(t+1)!} (\frac{\partial}{\partial x} + f(x, y)\frac{\partial}{\partial y})^t f(x_n, y_n) \qquad \text{(B.6)}$$

we define

$$D = \frac{\partial}{\partial x} + f_n \frac{\partial}{\partial y} \qquad f_n = f(x_n, y_n) \qquad \text{(B.7)}$$

for example, we can write:

$$[\frac{\partial}{\partial x}+f(\frac{\partial}{\partial y})]^2 f_n=\{D^2 f+f_y Df\}_n$$

By using Eq.(B.7), Eq.(B.6) becomes:

$$y_{n+1}=y_n+hf(x_n,y_n)+\{\frac{h^2}{2!}Df+\frac{h^3}{3!}(D^2 f+f_y Df)$$

$$+\frac{h^4}{4!}(D^3 f+f_y D^2 f+f_y^2 Df+3DfDf_y)+\frac{h^5}{5!}(D^4 f$$

$$+6DfD^2 f_y+4D^2 fDf_y+Dff_y^3+3(Df)^2 f_{yy}+D^3 ff_y$$

$$+7f_y DfDf_y)]\}_n+O(h_n^6) \qquad \text{(B.8)}$$

To get the expansion of R.H.S of Eq.(B.1), we first use the Taylor-series expansion for two variables to write:

$$f[x_n+\alpha_i h_n, y_n+(\sum_{j=1}^{i-1}\beta_{ij})h_n f_n]=\sum_{t=0}^{\infty}\frac{h_n^t}{t!}D_i^t f(x_n,y_n) \qquad \text{(B.9)}$$

where:

$$D_i=\alpha_i\frac{\partial}{\partial x}+(\sum_{j=1}^{i-1}\beta_{ij})f_n\frac{\partial}{\partial y}. \qquad \text{(B.10)}$$

By using Eqs.(B.9) and (B.10), we can get expansions for each k_i on the R.H.S of Eq.(B.1).

Since $\alpha_1=0$,

$$k_1 = h_n f_n \tag{B.11}$$

for k_2 we have

$$k_2 = h_n f(x_n + \alpha_2 h_n,\ y_n + \beta_{21} k_1) = h_n f(x_n + \alpha_2 h_n,\ y_n + \beta_{21} h_n f_n)$$

$$= h_n \sum_{t=0}^{\infty} h_n^t D_2^t f(x_n, y_n)/t! \tag{B.12}$$

where D_2 is given by Eq.(B.10).

For k_3 we proceed as follows using Eqs.(B.11,12):

$$k_3 = h_n f(x_n + \alpha_3 h_n,\ y_n + \beta_{31} k_1 + \beta_{32} k_2)$$

$$= h_n f[x_n + \alpha_3 h_n,\ y_n + (\beta_{31} + \beta_{32}) h_n f_n + \beta_{32}(k_2 - h_n f_n)]$$

$$= h_n \sum_{t=0}^{\infty} [h_n D_3 + \beta_{32}(k_2 - h_n f_n)(\partial/\partial y)]^t f(x_n, y_n)/t! \tag{B.13}$$

By using Eq.(B.12)in Eq.(B.13), we may consider Eq.(B.13) to be an expansion in powers of h_n. The procedure for k_3 suggests the general procedure for k_i. we write

$$k_i = h_n f(x_n + \alpha_i h_n, y_n + \sum_{j=1}^{i-1} \beta_{ij} k_j)$$

$$= h_n f\{(x_n + \alpha_i h_n, y_n + \sum_{j=1}^{i-1} [\beta_{ij} h_n f_n + \beta_{ij}(k_j - h_n f_n)]\}$$

$$= h_n \sum_{t=0}^{\infty} [h_n D_i + \sum_{j=1}^{i-1} \beta_{ij}(k_j - h_n f_n)\frac{\partial}{\partial y}]^t f(x_n, y_n)/t! \qquad \text{(B.14)}$$

and then use the results for k_j, $j < i$ to write k_i as an expansion in powers of h_n. The results for i=1,2,3,4, will be:

$$k_1=\{hf\}_n \qquad \text{(B.15)}$$

$$k_2=\{hf+h^2D_2f+(h^3/2!)D_2^2 f+(h^4/3!)\ D_2^3f+(h^5/4!)D_2^4\ f\}_n+O(h^6{}_n)$$

(B.16)

$$k_3 = \{ hf + h^2 D_3 f + h^3(\frac{1}{2}D_3^2 f + \beta_{32} f_y D_2 f) + h^4 [\frac{1}{6} D_3^3 f$$

$$+(\beta_{32}/2) f_y D_2^2 f + \beta_{32} D_2 f D_3 f_y] + h^5 [\frac{1}{24} D_3^4 f + (\beta_{32}/6) f_y D_2^3 f$$

$$+(\beta_{32}/2) D_2^2 f D_3 f_y + (\beta_{32}^2/2) f_{yy} (D_2 f)^2] + (\beta_{32}/2) D_2 f D_3^2 f_y]\}_n$$

$$+ O(h_n^6) \qquad \text{(B.17)}$$

$$k_4= hf+h^2D_4f+h^3[(1/2)\,D_2^4f+\beta_{42}f_yD_2f+\beta_{43}f_yD_3f]$$

$$+\ h^4[(1/6)\ D_4^3f+1/2\,\beta_{42}f_yD_2^2f+\beta_{32}\,\beta_{43}\ (f_y)^2D_2f+(1/2)\beta_{43}f_yD_3^2f$$

$$+\beta_{42}D_2fD_4f_y+\beta_{43}D_3fD_4f_y]+h^5[(1/24)\,D_4^4f+(1/6)\beta_{42}f_yD_2^3f$$

$$+(1/2)\beta_{32}\,\beta_{43}\ (f_y)^2\,D_2^2f+\beta_{32}\,\beta_{43}f_yD_2fD_3f_y+(1/6)\beta_{43}f_yD_3^3f$$

$$+(1/2)\beta_{42}D_4f_yD_2^2f+(1/2)\beta_{43}D_4f_yD_3^2f+(1/2)\beta_{42}^2f_{yy}D_2^2f$$

$$+\beta_{42}\,\beta_{43}f_{yy}D_2fD_3f+(1/2)\beta_{43}^2f_{yy}D_3^2f+(1/2)\beta_{42}D_2fD_4^2f_y$$

$$+1/2\beta_{43}D_3fD_4^2f_y+\beta_{43}\beta_{32}f_yD_2fD_4f_y]\}_n+O(h_n^6) \qquad \text{(B.18)}$$

Substituting Eq.(B.15) to Eq.(B.18) and Eq.(B.8) into Eq.(B.1), and matching powers of h_n through h_n^4 we get:

$$h_n: \quad w_1+w_2+w_3+w_4=1 \qquad \text{(B.19)}$$

$$h_n^2: \quad w_2D_2f+w_3D_3f+w_4D_4f=\frac{1}{2}Df \qquad \text{(B.20)}$$

$$h_n^3: \quad \frac{1}{2}[w_2D_2^2f+w_3D_3^2f+w_4D_4^2f]+f_y[w_3\beta_{32}D_2f$$

$$+w_4(\beta_{42}D_2f+\beta_{43}D_3f)]=\frac{1}{3!}(D^2f+f_yDf) \qquad \text{(B.21)}$$

$$h_n^4: \quad \frac{1}{3!}[w_2D_2^3f+w_3D_3^3f+w_4D_4^3f]+\frac{1}{2}f_y[w_3\beta_{32}D_2^2f$$

$$+w_4(\beta_{42}D_2^2f+\beta_{43}D_3^2f)]+[w_3\beta_{32}D_2fD_3f_y+w_4(\beta_{42}D_2fD_4f_y$$
$$+\beta_{43}D_3fD_4f_y)]+[w_4\beta_{32}\beta_{43}(f_y)^2D_2f]=(1/4!)[D^3f+f_yD^2f$$
$$+3DfD_y+f_y^2Df] \qquad \text{(B.22)}$$

If the valuse of $w_i's, \alpha_i's$ and $\beta_{ij}'s$ are to be independent of *f(x,y),* then the expansions in square brackets on the L.H.S. of Eqs.(B.21) and (B.22) must equal the corresponding terms on the R.H.S. Moreover, if the resulting eight equations are to be independent of *f(x,y),* then the ratio

$$D_j f / D f \quad j=2,3,4 \qquad \text{and} \qquad D_j f_y / D f_y \quad j=3,4 \tag{B.23}$$

must be constant. This will be true if

$$\alpha_i = \sum_{j=1}^{i-1} \beta_{ij} \qquad i=2,3,4 \tag{B.24}$$

for then

$$D_i = \alpha_i D \tag{B.25}$$

Finally, then the eight equations are

$$\begin{aligned}
& w_1 + w_2 + w_3 + w_4 = 1 \\
& w_2\alpha_2 + w_3\alpha_3 + w_4\alpha_4 = \frac{1}{2} \\
& w_2\alpha_2^2 + w_3\alpha_3^2 + w_4\alpha_4^2 = \frac{1}{3} \\
& w_3\alpha_2\beta_{32} + w_4(\alpha_2\beta_{42} + \alpha_3\beta_{43}) = \frac{1}{6} \\
& w_2\alpha_2^3 + w_3\alpha_3^3 + w_4\alpha_4^3 = \frac{1}{4} \\
& w_3\alpha_2^2\beta_{32} + w_4(\alpha_2^2\beta_{42} + \alpha_3^2\beta_{43}) = \frac{1}{12} \\
& w_3\alpha_2\alpha_3\beta_{32} + w_4(\alpha_2\beta_{42} + \alpha_3\beta_{43})\alpha_4 = \frac{1}{8} \\
& w_4\alpha_2\beta_{32}\beta_{43} = \frac{1}{24}
\end{aligned} \tag{B.26}$$

In third order Runge –Kutta method we need only the following equations:

$$w_1 + w_2 + w_3 = 1 \tag{B.27a}$$

$$w_2\alpha_2 + w_3\alpha_3 = 1/2 \tag{B.27b}$$

$$w_2\alpha_2^2 + w_3\alpha_3^2 = 1/3 \tag{B.27c}$$

$$w_3\alpha_2\beta_{32} = 1/6 \tag{B.27d}$$

$$\alpha_2 = \beta_{21} \tag{B.27e}$$

$$\alpha_3 = \beta_{31} + \beta_{32} \tag{B.27f}$$

To find w_3 , we have from Eq.(B.27c):

$$w_3 = \frac{1}{3\alpha_3^2} - \frac{\alpha_2^2}{\alpha_3^2} w_2$$

and by using Eqs.(B.27b)and(B.27d) we have:

$$w_3 = \frac{1}{3\alpha_3^2} - \frac{\alpha_2^2}{\alpha_3^2}\left(\frac{3\alpha_3 - 2}{6\alpha_2(\alpha_3 - \alpha_2)}\right)$$

therefore:

$$w_3 = \frac{2 - 3\alpha_2}{6\alpha_3(\alpha_3 - \alpha_2)} . \tag{B.28}$$

To find w_2 , we have from Eq.(B.27b):

$$w_2 = \frac{1}{2\alpha_3^2} - \frac{\alpha_3}{\alpha_2} w_3$$

and by using Eq.(B.28) , w_2 can be written as:

$$w_2 = \frac{3\alpha_3 - 2}{6\alpha_2(\alpha_3 - \alpha_2)}. \qquad \text{(B.29)}$$

To find w_1 , we will substitute from Eqs.(B.28)and(B.29) into Eq.(B.27a), then one can write w_1 as:

$$w_1 = 1 + \frac{2 - 3(\alpha_2 + \alpha_3)}{6\alpha_2\alpha_3}. \qquad \text{(B.30)}$$

To find β_{32}, from Eq.(B.27d) we have:

$$\beta_{32} = \frac{1}{6\alpha_2 w_3}$$

and by using Eq.(B.28), β_{32} can be written as:

$$\beta_{32} = \frac{\alpha_3(\alpha_3 - \alpha_2)}{\alpha_2(2 - 3\alpha_2)} . \qquad \text{(B.31)}$$

To find β_{31}, Eq.(B.24) gives:

$$\alpha_3 = \beta_{32} + \beta_{31}$$

therefore:

$$\beta_{31} = \alpha_3 - \beta_{32}$$

by using Eq.(B.31), β_{31} can be written as:

$$\beta_{31} = \frac{3\alpha_2\alpha_3(1-\alpha_2)-\alpha_3^2}{\alpha_2(2-3\alpha_2)}. \qquad \text{(B.32)}$$

Now returnning to Eq.(B.1), the second order Runge –Kutta method can be written as:

$$y_{n+1} = y_n + w_1k_1 + w_2k_2 \qquad \text{(B.33)}$$

where:

$$k_1 = h_n f(x_n, y_n)$$

$$k_2 = h_n f(x_n + \alpha_2 h_n, y_n + \beta_{21}k_1)$$

if $\alpha_2 = \frac{1}{2}$, and $\alpha_3 = 1$, then:

$$w_1 = 1 + \frac{2-3(\frac{1}{2}+1)}{6(\frac{1}{2})(1)} = \frac{1}{6}$$

$$w_2 = \frac{3(1)-2}{6(\frac{1}{2})(1-\frac{1}{2})} = \frac{2}{3}$$

$$k_2 = h_n f(x_n + \frac{1}{2}h_n, y_n + \frac{1}{2}k_1)$$

Therefore Eq.(B.33) can be rewritten as:

$$y_{n+1} = y_n + \frac{1}{6}k_1 + \frac{2}{3}k_2$$

so,

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 4k_2). \tag{B.34}$$

In our study , one can trace a ray through a medium to find the position vector of the ray at step $n+1$ in terms of r_n , r_{n+1}, by applying Eq.(B.34), (see chapter 2).

The third order Runge-Kutta method can be written as:

$$y_{n+1} = y_n + w_1 k_1 + w_2 k_2 + w_3 k_3$$

where:

$$w_3 = \frac{2 - 3(\frac{1}{2})}{6(1)(1 - \frac{1}{2})} = \frac{1}{6}$$

$$y_{n+1} = y_n + \frac{1}{6}k_1 + \frac{2}{3}k_2 + \frac{1}{6}k_3$$

$$= y_n + \frac{1}{6}(k_1 + 4k_2 + k_3) \tag{B.35}$$

In our study , one can trace a ray through a medium to find T_{n+1} by applying Eq.(B.35), (see chapter 2).where:

$$k_3 = h_n f(x_n + \frac{1}{2}h_n, y_n + (\beta_{31} + \beta_{32})k_2)$$
$$= h_n f(x_n + \frac{1}{2}h_n, y_n + (-1 + 2)k_2) = h_n f(x_n + \frac{1}{2}h_n, y_n + k_2)$$

Appendix C

THE RAY TRACING PROGRAM

FOR A LINEAR AXIAL GRIN.

```
C    This program is used to trace ray  through an axial GRIN lens
C   & to  compute the optical length.

      IMPLICIT REAL *8(A-H,N-Z)
      DATA PI,R0,DT/3.141,15,0.03/
      PARAMETER(CV1=0.05,CV2=0.00,D=3.0,N0=1.76,B0=-0.1136363)
      DIMENSION X(3,100),T(3,100),A(3,100),B(3,100),C(3,100),COF(4,3)
     &N1(3),A0(3),A2(3),N2(3),A3(3)

C INTRANCE PUPIL COORDINATE

      OPEN(6,FILE='1111')
      R0=15
      VR0(1)=0.0
      DO 1 K=1,16
      VR0(K)=0.1*(K-1)
  1   CONTINUE
      Z0=-1E9
      D1=ABS(Z0)

C VARYING THE FIELD ANGLE

      DO 2 IA=0,5,5
      AL=(IA-1)*5.0
      AL=AL*PI/180.0
  2   CONTINUE

C THE INTERSECTION OF RAY WITH FIRST SURFACE

      DO 3 IN=1,1
      X0=VR0(IN)
      R1=1.0/CV1
      R2=1.0E9
      Y0=0.0
      AP=SQRT(X0*X0+Y0*Y0)
      TN=SIN(AL)/COS(AL)
      Z0=0.0
      W=X0-Z0*TN
      Q1=1.0+TN*TN
      Q2=-2.0*(-W*TN+R1)
      Q3=X0*X0+Y0*Y0-2*X0*Z0*TN+Z0*Z0*TN*TN
      DET=SQRT(Q2*Q2-4*Q1*Q3)
```

```
      ZM=(-Q2-DET)/(2*Q1)
      ZP=(-Q2+DET)/(2*Q1)
      IF(R1.GT.1E6) THEN
      Z1=0.0
      ELSE IF(ABS(ZP).LT.ABS(ZM))  THEN
      Z1=ZP
      ELSE
      Z1=ZM
      ENDIF
      X1=W+Z1*TN
      Y1=0.0
      H1=SQRT(X1*X1+Y1*Y1)
      IF(H1.EQ.0.0) THEN
      CPS1=0.0
      SIP1=0.0
      ELSE
      CPS1= X1/H1
      SIP1=Y1/H1
      ENDIF
      X(1,1)= X1
      X(2,1)=Y1
      X(3,1)=Z1
      A0(1)=SIN(AL)
      A0(2)=0.0
      A0(3)=COS(AL)
      IF(R1.GT.0.0) THEN
      DZ1=+H1/SQRT(R1**2-H1**2)
      ELSE
      DZ1=-H1/SQRT(R1**2-H1**2)
      ENDIF
      DN=SQRT(1+DZ1**2)

C THE NORMAL TO THE FIRST SURFACE AT FIRST INTERSECTION POINT

      N1(1)=DZ1*CPS1/DN
      N1(2)=DZ1*SIP1/DN
      N1(3)=-1/DN
      CSP1=-(A0(1)*N1(1)+A0(2)*N1(2)+A0(3)*N1(3))
      XN1=N0*(1+B0*Z1)
      SNP1=SQRT(1.0-CSP1*CSP1)
      GM1=1/XN1
      OM1=+GM1*CSP1-SQRT(1-(GM1*SNP1)**2)

C THE DIRECTION COSINE OF THE REFRACTED RAY

      T(1,1)=XN1*(GM1*A0(1)+OM1*N1(1))
      T(2,1)=XN1*(GM1*A0(2)+OM1*N1(2))
      T(3,1)=XN1*(GM1*A0(3)+OM1*N1(3))

C ITERATION FOR GENERATING SUCCESSIVE POINTS ON THE RAY

      DO 4 J=1,100
      DO 5 I=1,2
      A(I,J)=0.0
      B(I,J)=0.0
      C(I,J)=0.0
      E(I,J)=0.0
  5    CONTINUE
      FC=DT*N0*N0*B0
      A(3,J)=FC*(1+B0*X(3,J))
      B(3,J)=FC*(1+B0*(X(3,J)+0.5*DT*T(3,J)+(1/8)*DT*A(3,J)))
      C(3,J)=FC*(1+B0*(X(3,J)+DT*T(3,J)+0.5*DT*B(3,J)))
      DO 17 I=1,3
      X(I,J+1)=X(I,J)+DT*(T(I,J)+(1/6)*(A(I,J)+2*B(I,J)))
  17   CONTINUE
      T(1,J+1)=T(1,1)
      NZ=N0*(1.0+B0*X(3,J+1))
      T(3,J+1)=SQRT(NZ**2-T(1,J+1)**2)
      TX=T(1,J+1)
      TZ=T(3,J+1)
      DET1=TX**2+TZ**2
      DET2=NZ**2
      XS=X(1,J+1)
```

```
      ZS=X(3,J+1)
      XS0=X(1,J)
      ZS0=X(3,J)
      SLOPE=(XS-XS0)/(ZS-ZS0)
      WRITE(6,333)ZS0,XS0
 333  FORMAT(2F12.5)

C INTERSECTION POINT WITH SURFACE

      IF(X(3,J+1).GE.D) GOTO 31
 4    CONTINUE
 31   ZS=X(3,J)
      XS=X(1,J)
      YS=X(2,J)
    3 CONTINUE
      CLOSE(6)
      END
```

Appendix D

THE RAY TRACING PROGRAM

FOR A QUADRATIC AXIAL GRIN.

```
C   This program is used to trace ray  through  a Quadratic axial GRIN lens
C  & to  compute the optical length.

      IMPLICIT REAL *8(A-H,N-Z)
      DATA PI,R0,DT/3.141,15,0.03/
      PARAMETER(CV1=1/39.96397,CV2=0.0,D=3.0,N0=1.76,B0=0.2)
      DIMENSION X(3,300),T(3,300),A(3,300),B(3,300),C(3,300),
      N1(3),A0(3),A2(3),N2(3),A3(3)

C INTRANCE PUPIL COORDINATE

      OPEN(6,FILE='2222')
      B01=-0.50*B0
      R0=15
      VR0(1)=0.0
      DO 1 K=1,16
      VR0(K)=0.1*(K-1)
  1   CONTINUE

C  VARYING THE FIELD ANGLE

      DO 2 IA=0,6,6
      AL=(IA-1)*5.0
      AL=AL*PI/180.0
  2   CONTINUE
      DO 3 IN=1,1

C  THE INTERSECTION OF RAY WITH FIRST SURFACE
      X0=VR0(IN)
      R1=1.0/CV1
      R2=1E9
      Y0=0.0
      AP=SQRT(X0*X0+Y0*Y0)
      TN=SIN(AL)/COS(AL)
      Z0=0.0
      W=X0-Z0*TN
      Q1=1.0+TN*TN
      Q2=-2.0*(-W*TN+R1)
      Q3=X0*X0+Y0*Y0-2*X0*Z0*TN+Z0*Z0*TN*TN
      DET=SQRT(Q2*Q2-4*Q1*Q3)
      ZM=(-Q2-DET)/(2*Q1)
      ZP=(-Q2+DET)/(2*Q1)
```

```
      IF(R1.GT.1E6) THEN
      Z1=0.0
      ELSE IF(ABS(ZP).LT.ABS(ZM))  THEN
      Z1=ZP
      ELSE
      Z1=ZM
      ENDIF
      X1=W+Z1*TN
      Y1=0.0
      H1=SQRT(X1*X1+Y1*Y1)
      IF(H1.EQ.0.0) THEN
      CPS1=0.0
      SIP1=0.0
      ELSE
      CPS1= X1/H1
      SIP1=Y1/H1
      ENDIF
      X(1,1)= X1
      X(2,1)=Y1
      X(3,1)=Z1
      A0(1)=SIN(AL)
      A0(2)=0.0
      A0(3)=COS(AL)
      IF(R1.GT.0.0) THEN
      DZ1=+H1/SQRT(R1**2-H1**2)
      ELSE
      DZ1=-H1/SQRT(R1**2-H1**2)
      ENDIF
      DN=SQRT(1+DZ1**2)

C THE NORMAL TO THE FIRST SURFACE AT FIRST INTERSECTION POINT

      N1(1)=DZ1*CPS1/DN
      N1(2)=DZ1*SIP1/DN
      N1(3)=-1/DN
      CSP1=-(A0(1)*N1(1)+A0(2)*N1(2)+A0(3)*N1(3))
      XN1=N0*(1+B0*Z1+B01*Z1**2)
      SNP1=SQRT(1.0-CSP1*CSP1)
      GM1=1/XN1
      OM1=+GM1*CSP1-SQRT(1-(GM1*SNP1)**2)

C THE DIRECTON COSINE OF THE REFRACTED RAY

      T(1,1)=XN1*(GM1*A0(1)+OM1*N1(1))
      T(2,1)=XN1*(GM1*A0(2)+OM1*N1(2))
      T(3,1)=XN1*(GM1*A0(3)+OM1*N1(3))
      TX=T(1,1)
      TZ=T(3,1)

C ITERATION FOR GENERATING SUCCESSIVE POINTS ON THE RAY

      DO 4 J=1,100
      DO 5 I=1,2
      A(I,J)=0.0
      B(I,J)=0.0
      C(I,J)=0.0
      E(I,J)=0.0
   5  CONTINUE
      FC=DT*N0**2
      TR1=B0+2.0*B01*X(3,J)
      TR2=1.0+B0*X(3,J)+B01*X(3,J)**2
      A(3,J)=FC*TR1*TR2
      TM=X(3,J)+0.5*DT*T(3,J)+(1/8)*DT*A(3,J)
      B(3,J)=FC*(1.0+B0*TM+B01*TM**2)*(B0+2.0*B01*TM)
      TMC=X(3,J)+DT*T(3,J)+0.5*DT*B(3,J)
      C(3,J)=FC*(1.0+B0*TMC+B01*TMC**2)*(B0+2.0*B01*TMC)
      DO 17 I=1,3
      X(I,J+1)=X(I,J)+DT*(T(I,J)+(1/6)*(A(I,J)+2*B(I,J)))
      T(I,J+1)=T(I,J)+(1/6)*(A(I,J)+4*B(I,J)+C(I,J))
  17  CONTINUE
      T(1,J+1)=T(1,1)
      NZ=N0*(1.0+B0*X(3,J+1)+B01*X(3,J+1)**2)
      T(3,J+1)=SQRT(NZ**2-T(1,J+1)**2)
```

```
      TX=T(1,J+1)
      TZ=T(3,J+1)
      DET1=TX**2+TZ**2
      DET2=NZ**2
      XS=X(1,J+1)
      ZS=X(3,J+1)
      ZS0=X(3,J)
      XS0=X(1,J)
      SLOP=(XS-XS0)/(ZS-ZS0)
      WRITE(6,111) ZS0,XS0
111   FORMAT(2F12.5)
      IF(X(3,J+1).GE.D)GOTO 31
4     CONTINUE
 31   ZS=X(3,J)
      XS=X(1,J)
  3   CONTINUE
      CLOSE(6)
      END
```

Appendix E

THE RAY TRACING PROGRAM

FOR A RADIAL GRIN.

```
C       This program is used to trace ray in  radial GRIN medium

        IMPLICIT REAL *8(A-H,N-Z)
        DIMENSION X(3,50),XC(3,500),XB(3,500),A(3,500),B(3,500),C(3,500)
        DIMENSION  A0(3),A1(3),A2(3),A3(3),N1(3),N2(3),T(3,500)
        DATA PI,G0,G1/3.14155926,1.76,-0.005/
        PARAMETER(CV1=0.05,CV2=0.0,D=3.0,DT=0.03,AL=20.0)
        open(6,file='3333')

C      ENTRANCE PUPIL COORDINATES

        X0=0.5
        Y0=0.0
        Z0=0.0
        TN=SIN(AL)/COS(AL)

C      INITIAL POINT & DIRECTION

        W=X0+Z0*TN
        Q1=1.0+TN*TN
        Q2=-2.0*(W*TN+1/CV1)
        Q3=W*W
        DET=SQRT(Q2*Q2-4*Q1*Q3)
        ZM=(-Q2-DET)/(2*Q1)
        ZP=(-Q2+DET)/(2*Q1)
        IF (ZP.LT.ZM) THEN
        Z1=ZP
        ELSE
        Z1=ZM
        ENDIF
        X1=W-Z1*TN
        Y1=0.0
        H1=SQRT(X1*X1+Y1*Y1)
        IF (H1.EQ.0.0) THEN
        CPS1=0.0
        ELSE
        CSP1=X1/H1
        ENDIF
        IF (H1.EQ.0.0) THEN
        SIP1=0.0
        ELSE
        SIP1=Y1/H1
```

```
      ENDIF
      X(1,1)= X1
      X(2,1)=Y1
      X(3,1)=Z1
      A0(1)=-SIN(AL)
      A0(2)=0.0
      A0(3)=COS(AL)
      DZ1=-H1/(Z1-1/CV1)
      DN=SQRT(1+(DZ1)**2)

C THE NORMAL TO THE FIRST SURFACE AT FIRST INTERSECTION POINT

      N1(1)=DZ1*CSP1/DN
      N1(2)=DZ1*SIP1/DN
      N1(3)=-1/DN
      CSP1=-(A0(1)*N1(1)+A0(2)*N1(2)+A0(3)*N1(3))
      XN1=G0+G1*(X1*X1+Y1*Y1)
      OM=CSP1-SQRT(XN1*XN1-(1-CSP1*CSP1))
      DO 10 I=1,3

C THE DIRECTION COSINE OF THE REFRACTED RAY

      T(I,1)=A0(I)+OM*N1(I)
   10 CONTINUE

C     ITERRATION FOR OTHER POINTS

      DO 20 J=1,100
      A(3,J)=0.0
      B(3,J)=0.0
      C(3,J)=0.0
      DO 21 I=1,2

C     A,B,C ARE MATRICES ELEMENTS

      NA=G0+G1*(X(1,J)**2+X(2,J)**2)
      S=2.0*G1
      A(I,J)=DT*XN1*S*X(I,J)
      XB(I,J)=X(I,J)+0.5*DT*T(I,J)+(1/8)*DT*A(I,J)
      NB=G0+G1*XB(1,J)**2+G1*XB(2,J)**2
      B(I,J)=DT*NB*S*XB(I,J)
      XC(I,J)=X(I,J)+DT*T(I,J)+0.5*DT*B(I,J)
      NC=G0+G1*(XC(1,J)**2+XC(2,J)**2)
      C(I,J)=DT*NC*S*XC(I,J)
   21 CONTINUE

C     GENERATE R ,T FOR NEXT POINT

      DO 22 I=1,3
      X(I,J+1)=X(I,J)+DT*(T(I,J)+(1/6)*(A(I,J)+2*B(I,J)))
   22 continue
      T(3,J+1)=T(3,1)
      T(2,J+1)=0.0
      T(1,J+1)=SQRT(NA**2-T(3,J+1)**2)
      XS=X(1,J+1)
      ZS=X(3,J+1)
      XS0=X(1,J)
      ZS0=X(3,J)
      SLOPE=(XS-XS0)/(ZS-ZS0)
      WRITE(6,333)ZS0,XS0
  333 FORMAT(2F12.5)
   33 CONTINUE
      close(6)
      END
```